The Chaotic Properties of the Shift Map and His Conjugacy of Horseshoe Map

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Abstract

In our study, we prove some chaotic properties of the shift map σ on the symbol space $\sum_2$ and apply the topological conjugacy property of the shift map on the horseshoe map.

Keywords: Devaney Chaos; Topological Conjugacy; Topological Mixing; Shift of finite type ; weakly blending; strongly blending .

1-Introduction

Chaotic dynamics generally refers to so complicated and seemingly random long term behavior exhibited in dynamical systems that keep simple, straightforward, deterministic laws . This type of dynamics can be seen in dynamical systems as various as electrical circuits, fluid dynamics, oscillating chemical reactions, and motion of planetary bodies .

The smale horseshoe map F is diffeomorphism defined on a square T in the plane . The image of $\mathrm{F}(T)$ is bound to form a horseshoe like shape [1]. From the axioms that map which are topologically conjugate are totally equivalent in terms of their dynamics . Particularly, the horseshoe map is topologically conjugate to the shift map σ . Hence the shift map is an exact model for the horseshoe map .

Let X be a compact metric space with no isolated point, $\mathcal{H}: \mathrm{X} \rightarrow \mathrm{X}$ be a map . In [2], Dzul-kifli and Good showed that the set of points with prime period at least n it is dense for each n if $\mathcal{H}$ is Devaney chaotic. In [3] Baloush and Dzul-kifli introduced six various one-step shift of finite type which have totally different dynamics demeanor and clear up the dynamics of each space . [4] showed that Locally Everywhere Onto implies many other chaos properties such as mixing , totally transitive ,and blending.

2. Preliminaries

If $\mathcal{H}: \mathrm{X} \rightarrow \mathrm{X}$ be a map on compact metric space with no isolated point, let $\mathrm{p} \in \mathrm{X}$ then $\mathcal{H}(\mathrm{p})$ = the first iterate of p for $\mathcal{H}$. More generally ,if n is any an integer , and $\mathrm{a_n}$ is the n-th iterate of p for $\mathcal{H}$, then $\mathcal{H}(\mathrm{a_n})$ is the $(\mathrm{n}+1)$ st iterate of p for $\mathcal{H}$. **The orbit** of p it is the set of points $\mathrm{p}, \mathcal{H}(\mathrm{p}), \mathcal{H}^2(\mathrm{p}), \ldots$, and is symbolize by $\mathrm{orb(p)} =$

$\{\mathcal{H}^n(p)|n \in N_\circ\}$ such that $N_\circ = N \cup \{0\}$. A point $p \in X$ it is said **a fixed point** of $\mathcal{H}$ if $\mathcal{H}(p) = p$, $\mathcal{H}$ is said to be topologically transitive if $\exists\, n > 0$ such that $\mathcal{H}^n(U) \cap V \neq \emptyset$, where U, V are any two non-empty open subsets of X . If $\exists\, \delta > 0$ for this any $x \in X$ and neighborhood N of x , $\exists\, y \in N$ and $n > 0$ where $d(\mathcal{H}^n(x), \mathcal{H}^n(y)) > \delta$, then $\mathcal{H}$ has sensitive dependence on initial conditions, briefly , we will write (SDIC). If for every pair non-empty open subsets U and V in X , there are a positive integer n such that $\mathcal{H}^k(U) \cap V \neq \emptyset$ for every $k > n$, then we called that $\mathcal{H}$ is topological mixing . If for any pair of non-empty open sets U and V in X , there exists some $n > 0$ such that $\mathcal{H}^n(U) \cap \mathcal{H}^n(V) \neq \emptyset$, then we called that $\mathcal{H}$ is weakly blending, , and called strongly blending, if for any pair of non-empty open sets U and V in X , there exists some $n > 0$ where $\mathcal{H}^n(U) \cap \mathcal{H}^n(v)$ contains a non-empty open subset. $\mathcal{H}$ is said to be *locally everywhere onto* if for every open set U of X , there exists a positive integer n such that $\mathcal{H}^n(U) = X$,[4] . Let $\mathcal{H}: A \to A$ and $\mathcal{L}: B \to B$ be two continuous map , if there exists a homeomorphism $h: A \to B$ such that $h^\circ\mathcal{H} = \mathcal{L}^\circ h$ then $\mathcal{H}$ and $\mathcal{L}$ are called a topologically conjugate . The homeomorphism h is said to be topological conjugacy between $\mathcal{H}$ and $\mathcal{L}$. [5]

3. On Various Concepts for Topological Dynamical Systems

The set $A_m = \{0,1\}$. we indicate to A_m as an alphabet and its elements as a symbols . Let $\sum_2$ ($\sum_2^+$) be the set of each bi-infinite sequences (1-sided sequences) with their elements of $\sum_2$ ($\sum_2^+$) ,i.e. every element $\mathcal{S}$ of $\sum_2$ ($\sum_2^+$) is of the form :

$\mathcal{S} = \{\dots, \mathcal{S}_{-i}, \dots, \mathcal{S}_{-1}, \mathcal{S}_0, \mathcal{S}_1, \dots, \mathcal{S}_i, \dots\}$, $\mathcal{S}_i \in A_m$, or $\mathcal{S} = \{\mathcal{S}_0, \mathcal{S}_1, \dots, \mathcal{S}_n, \dots\}$, $\mathcal{S}_i \in A_m$.

Now take another sequence $\overline{\mathcal{S}} \in \sum_2$, $\overline{\mathcal{S}} = \{\dots, \overline{\mathcal{S}}_{-i}, \dots, \overline{\mathcal{S}}_{-1}, \overline{\mathcal{S}}_0, \overline{\mathcal{S}}_1, \dots, \overline{\mathcal{S}}_i, \dots\}$, $\overline{\mathcal{S}}_i \in A_m$,

or $\overline{\mathcal{S}} = \{\overline{\mathcal{S}}_0, \overline{\mathcal{S}}_1, \dots, \overline{\mathcal{S}}_n, \dots\}$, $\overline{\mathcal{S}}_i \in A_m$. The metric between $\mathcal{S}$ and $\overline{\mathcal{S}}$ is defined as

$$d(\mathcal{S}, \bar{\mathcal{S}}) = \sum_{i=-\infty}^{\infty} \frac{1}{2^{|i|}} \quad , \quad \text{where } i \in \mathbb{Z} \text{ is the minimal number such that } \mathcal{S}_i \neq \bar{\mathcal{S}}_i$$

In case of bi-infinite sequences , or

$$d(\mathcal{S}, \bar{\mathcal{S}}) = \sum_{i=0}^{\infty} \frac{1}{2^i} \quad , \quad \text{where } i \in \mathbb{N} \text{ is the minimal number such that } \mathcal{S}_i \neq \bar{\mathcal{S}}_i$$

In case of 1-sided sequences . [6]

Definition 3.1 :

A shift of finite type (SFT) is a shift space $X \subset \sum_2$ which has a finite number of blocks from symbols 0 and 1 such that the blocks do not exist in any element of X . The blocks are called forbidden blocks in X . [3]

Definition 3.2 :

A shift of finite type (SFT) is an $M-$step or have memory M , for some integer $M \geq 1$, if it can be described by a set of forbidden blocks that have length $M+1$. [3]

Since we have four possible various blocks of length two i.e. 00,01,10 and 11 , then we have 16 sets of forbidden blocks, as shown in the table(1);

Table(1)
For each $i=\{1,2,\dots,16\}, X_i \subseteq \sum_2$ is the one-step SFT with set of forbidden blocks $\mathcal{F}_i$. [3]

$\mathcal{F}_1=\emptyset$	$\mathcal{F}_2=\{00\}$	$\mathcal{F}_3=\{01\}$	$\mathcal{F}_4=\{10\}$
$\mathcal{F}_5=\{11\}$	$\mathcal{F}_6=\{00,01\}$	$\mathcal{F}_7=\{00,10\}$	$\mathcal{F}_8=\{00,11\}$
$\mathcal{F}_9=\{01,10\}$	$\mathcal{F}_{10}=\{01,11\}$	$\mathcal{F}_{11}=\{10,11\}$	$\mathcal{F}_{12}=\{00,01,10\}$
$\mathcal{F}_{13}=\{00,01,11\}$	$\mathcal{F}_{14}=\{00,10,11\}$	$\mathcal{F}_{15}=\{01,10,11\}$	$\mathcal{F}_{16}=\{00,01,10,11\}$

4. Some Properties of the Shift Map

In this section we introduce the some results on shift of finite type space

Theorem 4.1

Let the one-step SFT $X_i, i=\{6,11,12,15\}$ is finite set then $\sigma: X_i \rightarrow X_i$ is stable

Proof :

If $X_i = X_6$ such that the forbidden is $\mathcal{F}_6=\{00,01\}$ since for every $\mathcal{S} \in X_6$, $\mathcal{S}_i \neq 0$ for every $i \in \mathbb{N}$, so $\mathcal{S}_i = 1$ for every $i \in \mathbb{N}$, so $\mathcal{S}_i=\{\overline{111}\}$ therefore X_6 is singleton set , so σ has the fixed point $\{\overline{111}\}$, and the basin of the fixed point is X_6 , so σ is stable .

In the same way $X_{11}=\{\overline{000}\}, X_{12}=\{\overline{111}\}, X_{15}=\{\overline{000}\}$. ■

Theorem 4.2 :

Let the one-step SFT X_i , $i=\{3,4\}$ is infinite set then $\sigma: X_i \rightarrow X_i, i=\{3,4\}$ is stable . Proof :

For every X_i , $i=\{3,4\}$ the forbidden of X_i , $i=\{3,4\}$ is $\mathcal{F}_3=\{01\}$ and $\mathcal{F}_4=\{10\}$, so X_i , $i=\{3,4\}$ has two fixed point and do not have any periodic point therefore the periodic point are not dense . Now let $\mathbb{U}=\{\overline{000}\}, \mathbb{V}=\{\overline{111}\}$ two open balls belong to X_i , $i=\{3,4\}$ then for any $\mathcal{S} \in \mathbb{U}$ $\sigma^n(\mathcal{S}) \notin \mathbb{V}$ for all integer n . So X_i , $i=\{3,4\}$ is not transitive and not Deveaney chaotic . Now if take the same open balls $\mathbb{U}$ and $\mathbb{V}$ such that $\sigma^n(\mathbb{U}) \cap \sigma^n(\mathbb{V}) = \emptyset$ for any integer n therefore X_i , $i = \{3,4\}$ is not weakly blending . To prove X_i , $i = \{3,4\}$ is not SDIC let $\mathcal{S} = \{\overline{00}\}$ and $\mathcal{T} = \{1\overline{00}\}$ then $d(\mathcal{S},\mathcal{T}) = 1$, $\sigma^n(\mathcal{S}) = \{\overline{00}\}$ and $\sigma^n(\mathcal{T}) = \{\overline{00}\}$ so $d\big(\sigma^n(\mathcal{S}),\sigma^n(\mathcal{T})\big) = 0$. Hence X_i , $i = \{3,4\}$ is not SDIC . ■

Theorem 4.3 :

Let the one-step SFT X_i , $i = \{7,9\}$ is finite set on discrete topology then $\sigma: X_i \to X_i$, $i = \{7,9\}$ have a weakness chaotic .

Proof :

If $X_i = X_7$ and $X_7 = \{\overline{111}, 0\overline{111}\}$ there are only four open balls are $X_7, \emptyset, (\overline{111})$ *and* $(0\overline{111})$. Because $\sigma(\overline{111}) \cap \sigma(0\overline{111}) = (\overline{111})$ so X_7 is strongly blending and weakly blending . Since $(0\overline{111})$ dose not contain any periodic point then the periodic points is not dense. X_7 is not transitive because $(0\overline{111}) \notin \sigma^n(\overline{111})$ for all integer n . Now let $\mathcal{S} = (0\overline{11})$ and $\mathcal{T} = (\overline{11})$ then $d(\mathcal{S},\mathcal{T}) = 1$, $\sigma^n(\mathcal{S}) = (\overline{11})$, $\sigma^n(\mathcal{T}) = (\overline{11})$ so $d\big(\sigma^n(\mathcal{S}), \sigma^n(\mathcal{T})\big) = 0$, therefore $\nexists\, \delta > 0$ such that $d\big(\sigma^n(\mathcal{S}),\sigma^n(\mathcal{T})\big) > \delta$, therefore X_7 is not SDIC .

If $X_i = X_9$ and $X_9 = \{\overline{00}, \overline{11}\}$ there are four open balls in X_9 are $X_9, \emptyset, (\overline{00}),$ *and* $(\overline{11})$. The periodic points of X_9 are dense since every point of X_9 is periodic . to prove it is not transitive , so that let $\mathbb{U} = (\overline{00})$ and , $\mathbb{V} = (\overline{11})$, then $\sigma^n(\mathbb{U}) = (\overline{00})$, and $\sigma^n(\mathbb{U}) \cap \mathbb{V} = \emptyset$, for all $n > 0$. Now let $\delta = 1$, and let $\mathcal{S} = (\overline{00}), \mathcal{T} = (\overline{11})$. So for each $n > 0$, $\sigma^n(\mathcal{S}) = \mathcal{S}$, $\sigma^n(\mathcal{T}) = \mathcal{T}$, and $d\big(\sigma^n(\mathcal{S}),\sigma^n(\mathcal{T})\big) \geq 1$, therefore X_9 is SDIC .To prove X_9 is not weakly blending and not strongly blending , let $\mathbb{U} = (\overline{00})$ and , $\mathbb{V} = (\overline{11})$. Since $\sigma^n(\mathbb{U}) = \mathbb{U}$ and $\sigma^n(\mathbb{V}) = \mathbb{V}$, then $\sigma^n(\mathbb{U}) \cap \sigma^n(\mathbb{V}) = \emptyset$, for all $n > 0$. ■

Theorem 4.4 :

The one-step SFT X_8 on discrete topology then $\sigma: X_8 \to X_8$ has Devaney chaotic .

Proof:

Since $X_8 = \{\overline{01}, \overline{10}\}$ so it has the only open balls $X_8, \emptyset, (\overline{01})$*and* $(\overline{10})$. The periodic points of X_8 are dense since every point is periodic point . Now let $\mathbb{U} = (\overline{01})$ and , $\mathbb{V} = (\overline{10})$. So for $n > 0$, it is either $\sigma^n(\mathbb{U}) = \mathbb{U}$ or $\sigma^n(\mathbb{U}) = \mathbb{V}$. If $\sigma^n(\mathbb{U}) = \mathbb{U}$ then $\sigma^{n+1}(\mathbb{U}) = \mathbb{V}$, and $\sigma^{n+1}(\mathbb{U}) \cap \mathbb{V} \neq \emptyset$, and if $\sigma^n(\mathbb{U}) = \mathbb{V}$, then $\sigma^n(\mathbb{U}) \cap \mathbb{V} \neq \emptyset$. therefore X_8 is transitive , so that σ has Devaney chaotic. ■

Remark 4.1: If X_8 on any another topology then it is stable .
Theorem 4.5 :
The one-step SFT X_2 is infinite set then $\sigma: X_2 \to X_2$ has Devaney chaotic, and mixing topological, totally transitive, locally everywhere onto, weakly blending and strongly blending.

Proof:

To prove that the periodic points of X_2 are dense , let $\epsilon > 0$ and $\mathcal{S} = (\mathcal{S}_0\mathcal{S}_1\mathcal{S}_2 \dots) \in X_2$. Choose n such that $\frac{1}{2^n} < \epsilon$, now let $\mathcal{T} = (\mathcal{T}_0\mathcal{T}_1\mathcal{T}_2 \dots)$ be another point such that $\mathcal{S}_i = \mathcal{T}_i$ for $i = 0,1,2, \dots, n$. Then $d(\mathcal{S}, \mathcal{T}) < \frac{1}{2^n}$, therefore the set of periodic point to be dense in X_2 we need to structure a periodic point within ϵ of $\mathcal{S}$. Let $\mathcal{T} = (\overline{\mathcal{S}_0\mathcal{S}_1\mathcal{S}_2 \dots \mathcal{S}_n 1})$ it is obvious that $\mathcal{T}$ is periodic point within ϵ of $\mathcal{S}$. So the periodic point are dense in X_2 .

To show that X_2 is locally everywhere onto let $\mathbb{U}$ be nonempty open ball in X_2 such that $\mathcal{S} = (\mathcal{S}_0\mathcal{S}_1\mathcal{S}_2 \dots \mathcal{S}_n \dots) \in \mathbb{U}$, then we have two statuses ; status 1: if $\mathcal{S}_n = 1$, because (10) and (11) are allowed , then $\sigma^n(\mathbb{U}) = X_2$. Status 2: if $\mathcal{S}_n = 0$, because (00) is forbidden then $\forall\, \mathcal{S} \in \mathbb{U}, \mathcal{S}_{n+1} = 1$ therefore $\sigma^{n+1}(\mathbb{U}) = X_2$. because for every open set $\mathcal{S} \subseteq X_2$ there exists a positive integer n such that $\sigma^n(\mathbb{U}) = X_2$, so that X_2 is locally everywhere onto .

Since X_2 is locally everywhere onto , therefore it is transitive , topological mixing , totally transitive , weakly blending and strongly blending . And since X_2 has dense of periodic point and transitive , then it is SDIC, therefore X_2 is Devaney chaotic . ■

5. The Horseshoe Map

The horseshoe map will be denoted by F. Its domain is the set S in $\mathbb{R}^2$ collected of the unit square $T = [0,1] \times [0,1]$, bounded on the left and right by semicircles B and E such that S contains its boundary . The map F shrinks S vertically by a factor of $a < {}^1\!/_3$, and expands S horizontally by a factor of $b = 3$. The result figure is folded by F therefore it fits again inside S , with only the semicircles popeyed to the left of T. Thus the range of F looks like a horseshoe when S is partitioned . We can see the effect of F on each member of the partition . Specifically, F sends semicircles B and E in to B and sends the square T into two strips inside T plus a curved strip inside E . [7]

The base interest in the horseshoe map F is to describe its dynamics on the attractor :

$$\Lambda = \{X \in T : F^n(X) \in T \ , \qquad \forall\ n \in \mathbb{Z}\}$$

To make our task easier , we first consider the set

$$\Lambda^{+} = \{\mathrm{X} : \mathrm{F}^{n}(\mathrm{X}) \in T \ , \ \forall \ n \in \mathbb{Z}^{+}\}$$

For the positive orbit of X , $orb^{+}(\mathrm{X})$. To be in T , X must belong to either V_0 or V_1 . Now if $\mathrm{F}^{n}(\mathrm{X}) \in T$, then obviously $\mathrm{F}^{n}(\mathrm{X}) \in V_0 \cup V_1$ or $\mathrm{X} \in F^{-1}(V_0) \cup F^{-1}(V_1)$. Now if deduce that Λ^{+} is the product of a cantor set with a vertical interval .

Next we take the set

$$\Lambda^{-} = \{\mathrm{X} : \mathrm{F}^{n}(\mathrm{X}) \in T \ , \ \forall \ n \in \mathbb{Z}^{-}\}$$

For the negative orbit of X , $orb^{-}(\mathrm{X})$. To be in T , X must belong to either $E_0 = F(V_0)$ or $E_1 = F(V_1)$. Now if $F^{-1}(\mathrm{X}) \in E_0 \cup E_1$, therefore $\mathrm{X} \in F(E_0) \cup F(E_1)$, (see Fig (1)) . [8]

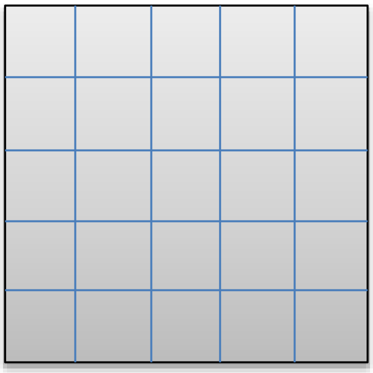

Figure (1)

6. Applications of topological conjugacy

Now , we define the topological conjugacy map $h : \sum_2 \to \Lambda$ is defined as follows :

For $\mathcal{S} \in \sum_2$ we let

$h(\mathcal{S}) = \{\mathcal{S}_0\mathcal{S}_2\mathcal{S}_1 \ldots\}$, where $\mathcal{S}_n = \begin{cases} \mathrm{H}^{n}(\mathcal{S}) \in V_0 & \mathcal{S}_n = 0 \ , \quad n \in \mathbb{Z}^{+} \\ \mathrm{H}^{n}(\mathcal{S}) \in V_1 & \mathcal{S}_n = 1 \ , \quad n \in \mathbb{Z}^{+} \end{cases}$

$h(\mathcal{S}) = \{\ldots\mathcal{S}_{-3}\mathcal{S}_{-2}\mathcal{S}_{-1}\}$, where $\mathcal{S}_n = \begin{cases} \mathrm{H}^{n}(\mathcal{S}) \in E_0 & \mathcal{S}_n = 0 \ , \quad n \in \mathbb{Z}^{-} \\ \mathrm{H}^{n}(\mathcal{S}) \in E_1 & \mathcal{S}_n = 1 \ , \quad n \in \mathbb{Z}^{-} \end{cases}$

Theorem 6.1 :

Let $h : \sum_2 \to \Lambda$ be a map , then h is a homeomorphism .

Proof :

To prove that h is one-to-one . Let $\mathcal{S}$ and $\mathcal{T}$ are in $\sum_2$, and $h(\mathcal{S}) = h(\mathcal{T})$, then $h(\mathcal{S})(h^{-1}(\mathcal{S}))$ and $h(\mathcal{T})$ $(h^{-1}(\mathcal{T}))$ lie on the same vertical (horizontal) line in T , such that thy have the same forward (backward) sequence . Therefore $\mathcal{S} = \mathcal{T}$, so that h is one-to-one .

To prove that h is onto , let $J_n = \{V\ in\ C_0 \cup C_1\ :\ h(\mathcal{S}_0\mathcal{S}_1\mathcal{S}_2 \dots) = V\}$ and $J_{-n} = \{V\ in\ C_0 \cup C_1\ :\ h(\dots \mathcal{S}_{-3}\mathcal{S}_{-2}\mathcal{S}_{-1}) = V\}$ then J_n and J_{-n} are closed for all n . Because $\bigcap_{n\geq 0} J_n$ is a single vertical line and $\bigcap_{n<0} J_n$ is single horizontal line in T . It follows that $\bigcap_{-\infty<n<\infty} J_n$ is a unique point V^* . By construction , $h(\mathcal{S}) = V^*$ such that $\mathcal{S} = \cdots \mathcal{S}_{-3}\mathcal{S}_{-2}\mathcal{S}_{-1}, \mathcal{S}_0\mathcal{S}_2\mathcal{S}_1 \dots$, in $\sum_2$ for all n . so that h is onto .

Therefore we need only to show that h and h^{-1} are continuous , let $\mathcal{S} = \cdots \mathcal{S}_{-2}\mathcal{S}_{-1}, \mathcal{S}_0\mathcal{S}_1\mathcal{S}_2 \dots$ and $\mathcal{T} = \cdots \mathcal{T}_{-2}\mathcal{T}_{-1}, \mathcal{T}_0\mathcal{T}_1\mathcal{T}_2 \dots$ be in $\sum_2$, with $h(\mathcal{S}) = V$ and $h(\mathcal{T}) = W$, if $d(\mathcal{S},\mathcal{T}) = \|\mathcal{S} - \mathcal{T}\| = \sum_{|k|=n+1}^{\infty} \frac{|\mathcal{S}_k - \mathcal{T}_k|}{2^{|k|}} \leq \frac{1}{2^{n-1}}$, then $\mathcal{S}_k = \mathcal{T}_k$ for $k = 0,1,2,\dots,n$, then V and W lie in the same vertical strip of width $1/_{3^{n+1}}$, so that $\|V - W\| < 1/_{3^{n+1}}$. similarly , $\mathcal{S}_k = \mathcal{T}_k$ for $k = -1,-2,\dots,-n$,which means that V and W lie in the same horizontal strip at the nth stage . So that there is a $\delta_1 > 0$ such that $\|V - W\| < \delta_1$. Now choose $\delta > 0$ such that $\delta < 1/_{3^{n+1}}$ and $\delta < \delta_1$ it follows that $d(V,W) = \|V - W\| < \delta$. consequently h is continuous . The proof that h^{-1} is continuous follows by a similar argument . ■

Proposition 6.2 [9]

1. d is a metric on $\sum_2$.
2. If $\mathcal{S}_i = \mathcal{T}_i$ for $i = 0,\dots,k$, then $d[\mathcal{S},\mathcal{T}] \leq 1/2^k$.
3. If $d[\mathcal{S},\mathcal{T}] < 1/2^k$ then $\mathcal{S}_i = \mathcal{T}_i$ for $i \leq k$.

Theorem 6.3 :

Let $I_1 = V_0 \cap E_0$, $I_2 = V_1 \cap E_0$, $I_3 = V_0 \cap E_1$ and $I_4 = V_1 \cap E_1$, Then $\bigcup_{j=1}^{4} I_j$ is closed and invariant under h.

Proof :

It is clear $h(I_j) \subset I_j$, $j = 1,2,3,4$ so $\bigcup_{j=1}^{4} I_j$ is invariant . To prove that $\bigcup_{j=1}^{4} I_j$ is closed, we need prove that each I_j , $j = 1,2,3,4$ is closed , we suppose that $\mathcal{S} \in I_j$, $j = 1,2,3,4$ such that $\mathcal{S} = \mathcal{S}_0\ \mathcal{S}_1\ \mathcal{S}_2 \dots$, $\mathcal{S}_i \in \{0,1\}$ for every $i \in \mathbb{N}$ which converge to $\mathcal{T}$. Let $\mathcal{T} \notin I_j$, $j = 1,2,3,4$. Since the $\mathcal{S}$ converge to $\mathcal{T}$, there is another integer k such that , if $i > k$ then $d(\mathcal{S},\mathcal{T}) \leq 1/2^{\alpha+1}$. By Proposition [4.3] , this forces $\mathcal{T}_0, \mathcal{T}_1, \dots, \mathcal{T}_{\alpha+1}$ to agree with the corresponding entries of $\mathcal{S}_i$ for $i \geq k$, so that $\mathcal{T}_\alpha \in \{0,1\}$ and $\mathcal{T} \in I_j$, $j = 1,2,3,4$, so $\bigcup_{j=1}^{4} I_j$ is closed . ■

Proposition 6.4 : The $h(\mathrm{X}_1)$ is located in $\bigcup_{j=1}^{4} I_j$.

Proof: since X_1 dose not have any forbidden block , then $h(\mathrm{X}_1)$ is located in $\bigcup_{j=1}^{4} I_j$.

Proposition 6.5 : The $h(X_2)$ is located in $I_2 \cup I_3 \cup I_4$.

Proof: Let $\mathcal{S} \in X_2$ such that for every $i \in \mathbb{Z}$, $\mathcal{S}_i = 0$, since $\{00\}$ is forbidden block, but 10 is allowed therefore $\mathcal{S}_{i-1} = 1$. and if $\mathcal{S}_i = 1$ then $\mathcal{S}_{i-1} = 1$ *or* 0 since 11 and 01 are allowed , so that $h(X_2) \notin I_1$, then $h(X_2)$ is located in $I_2 \cup I_3 \cup I_4$.

Proposition 6.6 : The $h(X_3)$ is located in $I_1 \cup I_3 \cup I_4$.

Proof: Let $\mathcal{S} \in X_3$ such that for every $\mathcal{S}_i = 1$ then $\mathcal{S}_{i-1} = 1$ for every $i \in \mathbb{Z}$ since 01 are forbidden , and if $\mathcal{S}_i = 0$ then $\mathcal{S}_{i-1} = 0$ *or* 1 since 00 and 10 are allowed , therefore $h(X_3) \notin I_2$ and $h(X_3)$ is located in $I_1 \cup I_3 \cup I_4$.

Proposition 6.7 : The $h(X_4)$ is located in $I_1 \cup I_2 \cup I_4$.

Proof: Let $\mathcal{S} \in X_4$ such that for every $\mathcal{S}_i = 0$ then $\mathcal{S}_{i-1} = 0$ for every $i \in \mathbb{Z}$ since 10 are forbidden , and if $\mathcal{S}_i = 1$ then $\mathcal{S}_{i-1} = 0$ *or* 1 since 00 and 01 are allowed, therefore $h(X_4) \notin I_3$ so $h(X_4)$ is located in $I_1 \cup I_2 \cup I_4$.

Proposition 6.8 : The $h(X_5)$ is located in $I_1 \cup I_2 \cup I_3$.

Proof: Since the only forbidden block is $\{11\}$ so that 01,10 and 00 are allowed , therefore for every $\mathcal{S} \in X_5$,if $\mathcal{S}_i = 1$ then $\mathcal{S}_{i-1} = 0$, $i \in \mathbb{Z}$, therefore $h(X_5) \notin I_4$ so $h(X_5)$ is located in $I_1 \cup I_2 \cup I_3$

Proposition 6.9 : The $h(X_6)$ is located in I_4 .

Proof: since the forbidden block of X_6 is $\{00{,}01\}$ then for every $\mathcal{S} \in X_6$, $\mathcal{S}_i \neq 0$ for every $i \in \mathbb{N}$. since 11 is allowed then $\mathcal{S}_i = 1$ for every $i \in \mathbb{Z}$, therefore $h(X_6) \notin I_1 \cup I_2 \cup I_3$ and $h(X_6)$ is located in I_4 .

Proposition 6.10 : The $h(X_7)$ is located in $I_2 \cup I_4$.

Proof: Since X_7 has two forbidden block $\{00\}$and $\{10\}$ then there is $\mathcal{S}_i = 1$ and $\mathcal{S}_{i-1} = 0$ *or* 1 , $i \in \mathbb{Z}$ such that $X_7 = \{\overline{111}, 0\overline{111}\}$ so that $h(X_7) \notin I_1 \cup I_3$ and $h(X_7)$ is located in $I_2 \cup I_4$.

Proposition 6.11 : The $h(X_8)$ is located in $I_2 \cup I_3$.

Proof: let $\mathcal{S} \in X_8$ such that for every $\mathcal{S}_i = 1$ then $\mathcal{S}_{i-1} = 0$ and if $\mathcal{S}_i = 0$ then $\mathcal{S}_{i-1} = 1$, $i \in \mathbb{Z}$, since $\{00\}$ and $\{11\}$ are forbidden block , so that $h(X_8) \notin I_1 \cup I_4$ and $h(X_8)$ is located in $I_2 \cup I_3$.

Proposition 6.12 : The $h(X_9)$ is located in $I_1 \cup I_4$.

Proof: since X_9 has two forbidden block $\{01\}$and $\{10\}$ then let $\mathcal{S} \in X_9$ such that for every $i \in \mathbb{Z}$ if $\mathcal{S}_i = 1$ than $\mathcal{S}_{i-1} = 1$ and if $\mathcal{S}_i = 0$ then $\mathcal{S}_{i-1} = 0$, so that $h(X_9) \notin I_2 \cup I_3$ and $h(X_9)$ is located in $I_1 \cup I_4$.

Proposition 6.13 : The $h(\mathrm{X}_{10})$ is located in $I_1 \cup I_3$.

Proof: since X_{10} has two forbidden block $\{01\}$and $\{11\}$ then let $\mathcal{S} \in \mathrm{X}_{10}$ such that for every $i \in \mathbb{Z}$ if $\mathcal{S}_i = 1$ than $\mathcal{S}_{i-1} = 0\ or\ 1$ such that $\mathrm{X}_{10} = \{\overline{000}, 1\overline{000}\}$, so $h(\mathrm{X}_{10}) \notin I_2 \cup I_4$ and $h(\mathrm{X}_{10})$ is located in $I_1 \cup I_3$.

Proposition 6.14 : The $h(\mathrm{X}_{11})$ is located in I_1 .

Proof: since the forbidden block of X_{11} is $\{10\}$ and $\{11\}$ then for every $i \in \mathbb{Z}$ let $\mathcal{S} \in \mathrm{X}_{11}$, $\mathcal{S}_i \neq 1$. since 00 is allowed then $\mathcal{S}_i = 0$ for every $i \in \mathbb{Z}$, therefore $h(\mathrm{X}_{11}) \notin I_2 \cup I_3 \cup I_4$ and $h(\mathrm{X}_{11})$ is located in I_1 .

Proposition 6.15 : The $h(\mathrm{X}_{12})$ is located in I_4 .

Proof: The forbidden block of X_{12} is $\{00\}$, $\{01\}$ and $\{10\}$ so that for every $\mathcal{S} \in \mathrm{X}_{12}$, $\mathcal{S}_i = 1$, $i \in \mathbb{Z}$, therefore $h(\mathrm{X}_{12}) \notin I_1 \cup I_2 \cup I_3$ and so $h(\mathrm{X}_{12})$ is located in I_4 .

Proposition 6.16 The $h(\mathrm{X}_{13})$, $h(\mathrm{X}_{14})$ and $h(\mathrm{X}_{16})$ are empty .

Proposition 6.17 : The $h(\mathrm{X}_{15})$ is located in I_1 .

Proof: The forbidden block of X_{15} is $\{01\}$, $\{10\}$ and $\{11\}$ so that for every $\mathcal{S} \in \mathrm{X}_{15}$, $\mathcal{S}_i = 0$, $i \in \mathbb{Z}$, therefore $h(\mathrm{X}_{15}) \notin I_2 \cup I_3 \cup I_4$ and so $h(\mathrm{X}_{15})$ is located in I_1 .

Let $\mathrm{M}_1 = \bigcup_{j=1}^{4} I_j$, $\mathrm{M}_2 = I_2 \cup I_3 \cup I_4$, $\mathrm{M}_3 = I_1 \cup I_3 \cup I_4$, $\mathrm{M}_4 = I_1 \cup I_2 \cup I_4$, $\mathrm{M}_5 = I_1 \cup I_2 \cup I_3$, $\mathrm{M}_6 = I_4$, $\mathrm{M}_7 = I_2 \cup I_4$, $\mathrm{M}_8 = I_2 \cup I_3$, $\mathrm{M}_9 = I_1 \cup I_4$, $\mathrm{M}_{10} = I_1 \cup I_3$, $\mathrm{M}_{11} = I_1$.

7. Some Chaotic Properties of the Shift Map

Theorem 7.1 :

Let the map $\sigma: \mathrm{X}_i \to \mathrm{X}_i , i = \{2,5\}$ is $h-$ conjugate to the map $\mathrm{F}: \mathrm{M}_i \to \mathrm{M}_i , i = \{2,5\}$, and σ has chaotic map in sense of Devaney , topologically mixing , totally transitive , weakly blending , strongly blending and locally everywhere onto then so F .

Proof:

To prove the set of periodic points in $\mathrm{M}_i , i = \{2,5\}$ is dense , let $\mathbb{U}$ be any open set of $\mathrm{M}_i , i = \{2,5\}$ and since that σ $h-$conjugates F , then $h^{-1}(\mathbb{U})$ is an open set of $\mathrm{X}_i , i = \{2,5\}$ and thus must contain a $p-$periodic point $\mathcal{S} \in \mathrm{X}_i , i = \{2,5\}$. Since $\mathcal{S} = \sigma^p(\mathcal{S})$, so that $h(\mathcal{S}) = h\big(\sigma^p(\mathcal{S})\big) = (\mathrm{F})^p\big(h(\mathcal{S})\big)$. So $h(\mathcal{S})$ is a $p-$periodic point of F . Furthermore , $h(\mathcal{S}) \in h\big(h^{-1}(\mathbb{U})\big) = \mathbb{U}$, and therefore the set of periodic points are dense in $\mathrm{M}_i , i = \{2,5\}$. To prove F has locally everywhere onto , let $\mathbb{U}$ be any open set in $\mathrm{M}_i , i = \{2,5\}$ then $h^{-1}(\mathbb{U})$ is an open set of $\mathrm{X}_i , i = \{2,5\}$.

Since σ is locally everywhere onto , there exists a positive integer n such that $\sigma^n\big(h^{-1}(\mathbb{U})\big) = \mathrm{X}_i \,, i = \{2,5\}$, so that $h\left(\sigma^n\big(h^{-1}(\mathbb{U})\big)\right) = (\mathrm{F})^n\left(h\big(h^{-1}(\mathbb{U})\big)\right) = (\mathrm{F})^n(\mathbb{U})$. Since h is one to one and onto then $(\mathrm{F})^n(\mathbb{U}) = \mathrm{M}_i \,, i = \{2,5\}$, So F has locally everywhere onto .Since F has locally everywhere onto then it is transitive ,topologically mixing , totally transitive, weakly blending and strongly blending . Also since Fhas dense periodic point and transitive then it is SDIC and F has Devaney chaotic . ■

Theorem 7.2 :

Let the map $\sigma\colon \mathrm{X}_i \to \mathrm{X}_i \,, i = \{3,4,6,11,12,15\}$ is $h-$ conjugate to the map $\mathrm{F}\colon \mathrm{M}_i \to \mathrm{M}_i \,, i = \{3,4,6,11\}$,so that F is stable .

Theorem 7.3 :

Let the map $\sigma\colon \mathrm{X}_i \to \mathrm{X}_i \,, i = \{7,10\}$ is $h-$ conjugate to the map $\mathrm{F}\colon \mathrm{M}_i \to \mathrm{M}_i \,, i = \{7,10\}$, so that F has weakly blending and strongly blending .

Proof:

It is sufficient to prove F has strongly blending . Let $\mathbb{U}$ and $\mathbb{V}$ be two open sets in $\mathrm{M}_i \,, i = \{7,10\}$. Since σ has weakly blending and strongly blending then $h^{-1}(\mathbb{U})$ and $h^{-1}(\mathbb{V})$ are open sets of $\mathrm{X}_i \,, i = \{7,10\}$ and thus $\sigma^n\big(h^{-1}(\mathbb{U})\big) \cap \sigma^n\big(h^{-1}(\mathbb{V})\big)$ contains an open set , so

$$h\left(\sigma^n\big(h^{-1}(\mathbb{U})\big)\right) \cap h\left(\sigma^n\big(h^{-1}(\mathbb{V})\big)\right)$$

$$= (\mathrm{F})^n\left(h\big(h^{-1}(\mathbb{U})\big)\right) \cap (\mathrm{F})^n\left(h\big(h^{-1}(\mathbb{V})\big)\right)$$

$$= (\mathrm{F})^n(\mathbb{U}) \cap (\mathrm{F})^n(\mathbb{V}) \text{ contains an open set also .}$$

Hence F has weakly blending and strongly blending . ■

Theorem 7.4 :

Let the map $\sigma\colon \mathrm{X}_8 \to \mathrm{X}_8$ is $h-$ conjugate to the map $\mathrm{F}\colon \mathrm{M}_8 \to \mathrm{M}_8$, than F is chaotic map in sense of Devaney .

Proof:

To prove that F is chaotic , we first prove that it is transitive . Let $\mathbb{U}$ and $\mathbb{V}$ be two open sets in M_8 and suppose that F $h-$ conjugates σ , then $h(\mathbb{U})$ and $h(\mathbb{V})$ are open sets in X_8 . Since σ is transitive , there exists $n \in \mathbb{Z}^+$ such that $\sigma^n\big(h(\mathbb{U})\big) \cap h(\mathbb{V}) \neq \emptyset$. Hence $h\big((\mathrm{F})^n(\mathbb{U})\big) \cap h(\mathbb{V}) \neq \emptyset$, so $(\mathrm{F})^n(\mathbb{U}) \cap \mathbb{V} \neq \emptyset$. Hence F is transitive .

To prove that the set of periodic points are dense in M_8 . , let $\mathbb{U}$ be any open set of M_8 and since that σ h −conjugates F , then $h^{-1}(\mathbb{U})$ is an open set of X_8 so there is a p −periodic point $\mathcal{S} \in X_8$. Since $\mathcal{S} = \sigma^p(\mathcal{S})$, so that $h(\mathcal{S}) = h(\sigma^p(\mathcal{S})) = (F)^p(h(\mathcal{S}))$. So $h(\mathcal{S})$ is a p −periodic point of F . Furthermore , $h(\mathcal{S}) \in h(h^{-1}(\mathbb{U})) = \mathbb{U}$, and therefore the set of periodic points are dense in M_8 , so that F is Devaney chaotic . ■

Theorem 7.5 :

Let the map $\sigma: X_9 \to X_9$ is $h-$ conjugate to the map $F: M_9 \to M_9$, then F has dense periodic points and has SDIC .

Proof:

By the same technique used in the previous proof , so that F has dense periodic points. To prove SDIC , let $\delta > 0$ and $\mathcal{S} \in M_9$ and N is neighborhood of $\mathcal{S}$, $\exists$ $\mathcal{T} \in N$ and suppose that F $h-$ conjugates σ , then $h(\mathcal{S}) \in X_9$ and $h(N)$ is neighborhood of $h(\mathcal{S})$. Since X_9 is SDIC then for all $n > 0$, $d\left(\sigma^n(h(\mathcal{S})), \sigma^n(h(\mathcal{T}))\right) > \delta_1$, hence $d\left(h((F)^n(\mathcal{S})), h((F)^n(\mathcal{T}))\right) > \delta_1$, $d\left(h^{-1}\left(h((F)^n(\mathcal{S}))\right), h^{-1}\left(h((F)^n(\mathcal{T}))\right)\right) > \delta_1$. Consequently , $d((F)^n(\mathcal{S}), (F)^n(\mathcal{T})) > \delta$, so F is SDIC . ■

Conclusions

- The map $F: M_i \to M_i$, $i = \{2,5\}$ has chaotic map in sense of Devaney , topologically mixing , totally transitive , weakly blending , strongly blending and locally everywhere onto .
- The map $F: M_i \to M_i$, $i = \{3,4,6,11\}$, is stable .
- The map $F: M_i \to M_i$, $i = \{7,10\}$ has weakly blending and strongly blending .
- The map $F: M_8 \to M_8$ has chaotic in sense of Devaney .
- The map $F: M_9 \to M_9$, has dense periodic points and has SDIC .

Conflict of Interests.
There are non-conflicts of interest

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الخلاصة

في هذا العمل, درسنا بعض الخصائص الفوضوية المختلفة لفضاء الضرب على دالة التزحيف. اوجدنا ترافقا تبولوجيا بين دالة التزحيف ودالة حدوة الحصان لنقل الخواص الفوضوية المدروسة على فضاء $\sum_2$.

الكلمات الدالة : فوضى ديفيني؛ الترافق التبولوجي ؛ تبولوجي ممزوج ؛ التزحيف في خطوة واحدة ؛خلط ضعيف ؛ خلط بقوة .