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Effect of Toxicant and Predator Harvesting on a Predator-prey Model with Modified Leslie-Gower

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تأثير حصاد المواد السامة والحيوانات المفترسة على نموذج الفريسة مع تعديل ليزلي-جاور

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ABSTRACT

In this paper, the effect of pollutant in environment on prey and predator species and predator harvesting is modeled using modified Leslie-Gower and nonlinear harvesting rate. Positivity and bounded of solution are proved, conditions which make the model permanent is determined, existence and locally asymptotically stable for each of possibly equilibrium point is studied, numerical solution of proposed model is done to show the effect and to confirm the analytical result.

Key words:

Harvesting, Leslie-Gower model, Permanence, Stability analysis, Toxicant

الخلاصة

في هذا البحث، تم نمذجة تأثير الملوثات في البيئة على أنواع الفرائس والحيوانات المفترسة وحصاد الحيوانات المفترسة باستخدام معدل الحصاد المعدل ليزلي– جاور وغير الخطي. يتم إثبات الإيجابية وحدود الحل، ويتم تحديد الشروط التي تجعل النموذج دائمة، ويتم دراسة الوجود والاستقرار التقريبي محليًا لكل نقطة توازن محتملة، ويتم إجراء الحل العددي للنموذج المقترح، لإظهار التأثير وتأكيد النتيجة التحليلية

الكلمات المفتاحية: الحصاد، نموذج ليزلي– جاور، النموذج دائمة، والاستقرار التقريبي، المواد السامة

INTRODUCTION

Leslie and Gower [1] formulate the Leslie-Gower predator Model, the carrying capacity predator population in their model is proportional to the number of preys population, this principle is not recognized in the Lotka-Volterra Model [2]. In case of severe scarcity, the predator population can switch over to another population but their growth will be limited by the fact that their most favorite food is not available in a bounce, therefore, Aziz-Aloui and Daher [3] modified the Leslie-Gower Model and their model can be written as follows:

$$\frac{dx}{dt} = r_1 x \left(1 - \frac{x}{k}\right) - \frac{\alpha x}{1+\alpha T x} y$$

$$\frac{dy}{dt} = r_2 y \left(1 - \frac{y}{x+c}\right) \qquad (1)$$

Where x (t) and y (t) are numbers for prey and predator, respectively. In the absence of predator prey species x (t) grows with intrinsic growth rate r_1 and carring capacity k. r_2 is predator interstice growth rate, c is the additional food rate which provides protection to the predators and the predators consumes the prey species according Hollying Type functional responses [4]. Like working on modification of Lotka-Voltera model by incorporating many factors [5-7], many researchers considered the system.1 with taking into account to the many factors like time-delay, harvesting prey, harvesting predator, and infectious disease in populations [7-8]. Arkan and Shilan [9], studied the effect of all the harvesting factor, spreading disease and prey refigure on the system.1 included at most two factors therefore, in this paper Leslie-Gower model incorporating harvesting predator and toxicant, is considered and the dynamic behavior of the proposed model is studied. This paper is organized as follows: In the next section, the details of the assumptions in the new model and the significant of the parameters used in it, is discussed, the third section, deals with positive, boundedness of the model, in section four permanence of the model under certain condition is proved. In fifth section, all possible equilibrium points and their existence criteria are considered, stability analysis is presented in sixth section. Finally, the numerical verification of the analytical finding is done using MATLAB program.

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Materials and Methods

The mathematical Model

Suppose that there exists toxicant in the environment affect negatively on the growth of population (prey as well as predator) in system 1, and only predator species is economically important, and they are harvested with non-linear harvesting rate. There for if $u(t)$ is the toxicant concentration in the populations at time t, $w(t)$ is the environment concentration of toxicant at time t, consequently the dynamics of Leslie-Gower model with existence of pollutants and predator harvesting can be described by the following set of non-linear differential equation.

$$\begin{aligned}
\frac{dx}{dt} &= r_1 x\left(1-\frac{x}{k}\right)-\frac{\alpha x}{1+\alpha T x}y-\sigma_1 x u \\
\frac{dy}{dt} &= r_2 y\left(1-\frac{y}{x+c}\right)-\frac{qEy}{s_1 E+s_2 y}-\sigma_2 y u \\
\frac{dw}{dt} &= P+\sigma_3(x+y)u-\sigma_4 w-\sigma_5(x+y)w \qquad (2) \\
\frac{du}{dt} &= \sigma_5(x+y)w-\sigma_3(x+y)u-\sigma_6 u
\end{aligned}$$

Here the new parameters described in table 1.

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Table 1 Biological interpretation of parameters

Parameters	Description
P	Environmental exogenous input rate of toxicant
σ_4	Natural depletion rate of environment toxicant.
σ_6	Natural washout rate of toxicant from organism.
σ_5	Uptake of toxicant by population organism.
σ_1, σ_2	Rates at which prey and predator decreasing due to toxicant.
q	Catch ability coefficient to harvest the individual.
E	Effort applied to harvest the individuals.
s_1, s_2	Are suitable positive constants.
σ_3	Rate of dissemination of toxicant by population organism.

Preliminaries

In this section, some preliminaries result on the solutions of system 2 are proved

Theorem 1. The positive int. R_+^4 is forward invariant for system.2.

Proof: Suppose $(x(t), y(t), w(t), u(t))$ is solution of system2 that initiated in int. R_+^4, then

$\frac{dx}{dt} = 0$ at $\big(0, y(t), w(t), u(t)\big)$ and $\frac{dy}{dt} = 0$ at $\big(x(t), 0, w(t)\ u(t)\big)$

$\frac{dw}{dt} > 0$ at $\big(x(t), y(t), 0, u(t)\big)$ and $\frac{du}{dt} \geq 0$ at $(x(t), y(t), w(t), 0)$

So, each compartment of the system 2 cannot intersect the axis and hence, the solution remains positive, that is R_+^4 is forward invariant.

Theorem 2. All the solution of the system 2 that initiate R_+^4 are uniformly bounded.

Proof: Let $(x(t), y(t), w(t), u(t))$ be any solution of system.2 with positive condition $(x(0), y(0), w(0), u(0))$.

From the first and second equation of the system2, it is obtained that:

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$$\frac{dx}{dt} \leq r_1 x\left(1 - \frac{x}{k}\right)$$
$$\frac{dy}{dt} \leq r_2 y\left(1 - \frac{y}{x + c}\right)$$

Thus,

$$\lim_{t\to\infty} Sup(x(t) \leq k$$
$$\lim_{t\to\infty} Sup(y(t)) \leq k + c$$

Thus, for any $\epsilon > 0$, sufficiently small there $m_1 > 0$ such that:

$$x(t) \leq k + \epsilon$$
$$y(t) \leq k + c + \epsilon \qquad \forall\, t > m_1$$

And, from third and fourth equation of the system (2) it is obtained that

$$\frac{d(w + u)}{dt} = P - \sigma_4 w - \sigma_6 u$$
$$\leq P - \sigma(w + u)$$

Where $\sigma = \min\{\sigma_4, \sigma_6\}$

Then by solution the differential equation the following inequality verified:

$$\lim_{t\to\infty} Sup(w + u) \leq \frac{P}{\sigma}$$

Thus, there exist$m_2 > 0$, for any $\epsilon > 0$ such that:

$$w + u \leq \frac{P}{\sigma} + \epsilon \qquad \forall\, t > m_2$$

So, for any $\epsilon > 0$ (Sufficiently small): the following inequality obtained.

$$x(t) \leq k + \epsilon$$
$$y(t) \leq k + c + \epsilon \qquad \forall\, t > m$$
$$w + u \leq \frac{P}{\sigma} + \epsilon$$

Where $m = \max\{m_1, m_2\}$

The proof is complete.

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Permanence of the model

In ecology, determining the criteria which makes the model to be permanent is important, because it implies that the population continuous to exist. There for in this section it is proved that system2 is permanent under contain condition.

Definition [10], system2 is said to be permanent of there exist positive constants $k_1$and k_2 such that:

$$k_2 \geq \max\{\lim_{t\to\infty} Sup\, x(t), \lim_{t\to\infty} Sup\, y(t), \lim_{t\to\infty} Sup\, w(t), \lim_{t\to\infty} Sup\, u(t)\}$$
$$\geq \min\{\lim_{t\to\infty} \inf x(t), \lim_{t\to\infty} \inf y(t), \lim_{t\to\infty} \inf w(t), \lim_{t\to\infty} \inf u(t)\} \geq k_1$$

Theorem 3. If the following condition hold, then the system2 is permanent.

$$\alpha(k+c) + \sigma_1 \frac{P}{\sigma} < 1 \qquad (3)$$

$$\frac{q}{s_1} + \sigma_2 \frac{P}{\sigma} < 1 \qquad (4)$$

Where $\sigma = \min\{\sigma_4, \sigma_6\}$

Proof: From theorem2,

$$\lim_{t\to\infty} Sup(x(t) \leq k\,, \lim_{t\to\infty} Sup(y(t)) \leq k + c \text{ and } \lim_{t\to\infty} Sup(w+u) \leq \frac{P}{\sigma}$$

Thus, $\max\{\lim_{t\to\infty} Sup\, x(t), \lim_{t\to\infty} Sup\, y(t), \lim_{t\to\infty} Sup\, w(t), \lim_{t\to\infty} Sup\, u(t)\} \leq \max\{k+c, \frac{P}{\sigma}\}$

if $t \to \infty$, then

$$\frac{dx}{dt} \geq r_1 x \left(1 - \alpha(k+c) - \sigma_1 \frac{P}{\sigma} - \frac{x}{k}\right)$$

Thus, solving differential inequality, get

$.\lim_{t\to\infty} \inf x(t) \geq \left(1 - \alpha(k+c) - \sigma_1 \frac{P}{\sigma}\right) k = m_1 > 0$, due to condition. 3

$$\frac{dy}{dt} \geq r_2 y \left(1 - \frac{q}{s_1} - \sigma_2 \frac{P}{\sigma} - \frac{y}{m_1 + c}\right)$$

Thus, $\lim_{t\to\infty} \inf y(t) \geq (k + m_1)\left(1 - \frac{q}{s_1} - \sigma_2 \frac{P}{\sigma}\right) = m_2 > 0$, due to condition.4.

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$$\frac{dw}{dt} \geq P - \sigma_4 w - \sigma_5(2k+c)\, w$$

So, $\liminf_{t\to\infty} w(t) \geq \frac{P}{\sigma_5(2k+c)+\sigma_4} = m_3 > 0$

Now

$$\frac{du}{dt} \geq \sigma_5(x+y)w - \sigma_3(2k+c)u - \sigma_6 u$$

So, $\liminf_{t\to\infty} u(t) \geq \frac{\sigma_5(m_1+m_2)m_3}{\sigma_3(2k+c)+\sigma_6} = m_4 > 0$

Thus,

$$\{\liminf_{t\to\infty} x(t), \liminf_{t\to\infty} y(t), \liminf_{t\to\infty} w(t), \liminf_{t\to\infty} u(t)\} \geq m$$

Where $m = \min\{m_1, m_2, m_3, m_4\}$

The proof is complete.

Existence of Equilibriums

System1 has at most four non-negative equilibrium points:

1.1 Population free equilibrium point $E_1 = \left(0,0,\frac{P}{\sigma_4},0\right)$ exist for all parametric value.

1.2 Predator free equilibrium point is $E_2 = (x_2, 0, w_2, u_2)$, were

$w_2 = \frac{P+\sigma_3 x_2 u_2}{\sigma_4+\sigma_5 x_2}$, $x_2 = k\left(1 - \frac{\sigma_1}{r_1}u_2\right)$ and u_2 is the root for the following function:

$f(u) = a_1 u^2 - a_2 u + k\sigma_5 T$, with

$a_1 = \frac{k\sigma_1}{r_1}(\sigma_3\sigma_4 + \sigma_5\sigma_6) > 0$ and $a_2 = \frac{\sigma_1\sigma_5 Pk}{r_1} + \frac{r_1}{\sigma_1}a_1 + \sigma_4\sigma_6 > 0$

Since $f(0) = k\sigma_5 P > 0$ and $f\left(\frac{r_1}{\sigma_1}\right) = -\frac{\sigma_3\sigma_4 r_1}{\sigma_1} < 0$ so, by intermediate value theorem, $f(u)$ has a positive root in the interval $\left(0, \frac{r_1}{\sigma_1}\right)$ and hence

$x_2 = k\left(1 - \frac{\sigma_1}{r_1}u_2\right) > 0.$

Consequently, predator free equilibrium point exists for all parametric value.

1.3 Prey free equilibrium point is $E_3 = (0, y_3, w_3, u_3)$, were

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$$u_3 = \frac{1}{\sigma_2}\left(r_2 - \frac{r_2}{c}y_3 - \frac{qE}{s_1E+s_2y_3}\right)$$

$$w_3 = \frac{P+\sigma_3 y_3 u_3}{\sigma_4+\sigma_5 y_3}$$

And y_3 is the root for the following function

$$g(y) = \sigma_5 Py - \frac{1}{\sigma_2}\left((\sigma_3\sigma_4+\sigma_5\sigma_6)y+\sigma_4\sigma_6\right)\left(r_2 - \frac{r_2}{c}y - \frac{qE}{s_1E+s_2y}\right)$$

$g(0) = -\frac{\sigma_4\sigma_6}{\sigma_2}\left(r_2 - \frac{q}{s_1}\right)$ and $g(c) = \sigma_5 Tc + \frac{1}{\sigma_2}\left((\sigma_3\sigma_4+\sigma_5\sigma_6)y+\sigma_4\sigma_6\right)\left(\frac{qE}{s_1E+s_2c}\right) > 0$

$g(0) < 0$ If the following condition hold

$$r_2 > max\left\{\frac{q}{s_1}, \frac{qE}{(s_1E+s_2y_3)\left(1-\frac{y_3}{c}\right)}\right\} \tag{5}$$

So, intermediate value theorem guaranties that $g(y)$ has a positive root in the interval$(0,c)$. Consequently, prey free equilibrium exists if condition 5 hold

1.4 Interior equilibrium point is $E_4 = (x_4, y_4, w_4, u_4)$

Were, $w_4 = \frac{1}{\sigma_4}(P - \sigma_6 u_4) > 0$

$$u_4 = \frac{\sigma_5(x_4+y_4)P}{(\sigma_5\sigma_6+\sigma_3\sigma_4)(x_4+y_4)+\sigma_4\sigma_6}$$

And $x_4$and y_4 are solution of the following system of equation

$$r_1\left(1-\frac{x}{k}\right) - \frac{\alpha y}{1+\alpha Tx} - \frac{\sigma_1\sigma_5(x+y)P}{(\sigma_5\sigma_6+\sigma_3\sigma_4)(x+y)+\sigma_4\sigma_6} = 0 \tag{6a}$$

$$r_2\left(1-\frac{y}{x+c}\right) - \frac{qE}{s_1E+s_2y} - \frac{\sigma_2\sigma_5(x+y)P}{(\sigma_5\sigma_6+\sigma_3\sigma_4)(x+y)+\sigma_4\sigma_6} = 0 \tag{6b}$$

In eq.6a, if $x \to 0$ then $y \to \bar{y} = \frac{-a_3+\sqrt{a_3^2+4r_1\sigma_4\sigma_6\alpha(\sigma_5\sigma_6+\sigma_3\sigma_4)}}{2\alpha(\sigma_5\sigma_6+\sigma_3\sigma_4)} > 0$

and if $y \to 0$ then $x \to \bar{x} = \frac{-a_4+\sqrt{a_4^2+4\frac{r_1^2\sigma_4\sigma_6(\sigma_5\sigma_6+\sigma_3\sigma_4)}{k}}}{2\alpha(\sigma_5\sigma_6+\sigma_3\sigma_4)} > 0$

were, $a_3 = \sigma_1\sigma_5 P + \alpha\sigma_4\sigma_6 - r_1(\sigma_5\sigma_6+\sigma_3\sigma_4)$

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and $a_4 = \sigma_1\sigma_5 P + \frac{r_1\sigma_4\sigma_6}{k} - r_1(\sigma_5\sigma_6 + \sigma_3\sigma_4)$

In equ.6b, as $y \to 0$ then $x \to \tilde{x} = \frac{\sigma_4\sigma_6\left(r_2 - \frac{q}{s_1}\right)}{\sigma_2\sigma_5 T - (\sigma_5\sigma_6 + \sigma_3\sigma_4)\left(r_2 - \frac{q}{s_1}\right)} > 0$ if the following holds

$$\frac{q}{s_1} + \frac{\sigma_2\sigma_5 P}{\sigma_5\sigma_6 + \sigma_3\sigma_4} > r_2 > \frac{q}{s_1} \tag{7}$$

and if $x \to 0$ then $y \to \tilde{y}$ where $\tilde{y}$ is the positive root of the following equation

$$\frac{r_2}{c} s_2(\sigma_5\sigma_6 + \sigma_3\sigma_4)y^3 + a_5 y^2 + a_6 y - \left(r_2 - \frac{q}{s_1}\right) s_1\sigma_2\sigma_4 = 0$$

With

$$a_5 = \left(\frac{r_2}{c} s_1 E - r_2 s_2\right)(\sigma_5\sigma_6 + \sigma_3\sigma_4) + \frac{r_2}{c}\sigma_4\sigma_6 s_2 + \sigma_1\sigma_5 s_2 P$$

$$a_6 = \sigma_5\sigma_6 s_1 E_1 P + \left(\frac{r_2}{c} s_1 E - r_2 s_2\right)\sigma_4\sigma_6 - \left(r_2 - \frac{q}{s_1}\right)(\sigma_5\sigma_6 + \sigma_3\sigma_4) s_1 E$$

So, if the following condition holds

$$(\bar{x} > \tilde{x} \text{ and } \bar{y} < \tilde{y}) \text{ or } (\bar{x} < \tilde{x} \text{ and } \bar{y} > \tilde{y}) \tag{8}$$

then the system has positive solution (x_4, y_4).

Consequently, interior equilibrium point exists, if in addition to conditions 7 and 8, the following condition hold

$$P > \sigma_6 u_4 \tag{9}$$

Stability analysis

In this section, locally stability analysis of each non negative equilibrium points is studied as follows:

Jacobian matrix of the system 2 at E_1 is given as

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$$J(E_1)=\begin{bmatrix} r_1 & 0 & 0 & 0 \\ 0 & r_2-\frac{q}{s_1} & 0 & 0 \\ -\sigma_5\frac{P}{\sigma_4} & -\sigma_5\frac{P}{\sigma_4} & -\sigma_4 & 0 \\ \sigma_5\frac{P}{\sigma_4} & \sigma_5\frac{P}{\sigma_4} & 0 & -\sigma_6 \end{bmatrix}$$

Eigen values of $J(E_1)$ are $\lambda_{11}=r_1>0$,$\lambda_{12}=r_2-\frac{q}{s_1}\lambda_{13}=-\sigma_4<0$ and$\lambda_{14}=-\sigma_6<0$.

Hence E_1 is saddle with $\dim W^s(E_1)=3$ and $dimW^u(E_1)=1$ if $r_2<\frac{q}{s_1}$ but $\dim W^s(E_1)=dimW^u(E_1)=2$ if $r_2>\frac{q}{s_1}$

Jacobian matrix of the system 2 at E_2 is given as

$$J(E_2)=\begin{bmatrix} r_1\left(1-\frac{2x_2}{k}\right)-\sigma_1u_2 & -\frac{\alpha x_2}{1+\alpha Tx_2} & 0 & -\sigma_1x_2 \\ 0 & r_2-\frac{q}{s_1}-\sigma_2u_2 & 0 & 0 \\ \sigma_3u_2-\sigma_5w_2 & \sigma_3u_2-\sigma_5w_2 & -\sigma_4-\sigma_5x_2 & \sigma_3x_2 \\ \sigma_5w_2-\sigma_3u_2 & \sigma_5w_2-\sigma_3u_2 & \sigma_5x_2 & -\sigma_3x_2-\sigma_6 \end{bmatrix}$$

One of the eigen values of $J(E_2)$ is $\lambda_1=r_2-\frac{q}{s_1}-\sigma_2u_2$ and all other eigen values $\lambda_2,\lambda_3$and λ_4 are roots for the following equation

λ_4 are roots for the following equation

$$\lambda^3+A_1\lambda^2+A_2\lambda+A_3=0$$

$$A_1=a_{7+}a_8,\ \ A_2=a_7a_8+a_9+a_{10}, A_3=a_8a_9+\sigma_4a_{10}$$

With $a_7=\sigma_4+\sigma_6+(\sigma_3+\sigma_5)x_2$

$$a_8=\sigma_1u_2+\frac{2r_1x_2}{k}-r_1$$

$$a_9=\sigma_4\sigma_6+(\sigma_3\sigma_4+\sigma_5\sigma_6)x_2$$

$$a_{10}=(\sigma_5w_2-\sigma_3u_2)\sigma_1x_2$$

$$A_1A_2-A_3=a_8(a_7^2-a_9)+a_{10}(a_8+a_7-\sigma_4)+a_8a_{10}+a_7a_8^2+a_7a_9+a_8a_9$$

If the following condition holds

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$a_8 > 0$ And $a_{10} > 0$ (10)

If condition 10 holds, then all Routh-Hurwitz criteria $A_1 > 0, A_3 > 0$ and $A_1A_2 > A_3$ are satisfied, and hence λ_2, λ_3 and λ_4 have negative real part. Consequently, if in addition to condition.10, the following condition hold

$$r_2 < \frac{q}{s_1} + \sigma_2 u_2 \quad (11)$$

Conditions hold, then the system is locally asymptotically stable around the predator free equilibrium point

Jacobian matrix of the system 2 at E_3 is given as

$$J(E_3) = \begin{bmatrix} r_1 - \alpha y_3 - \sigma_1 u_3 & 0 & 0 & 0 \\ \frac{r_2 y_2^2}{c^2} & r_2\left(1 - \frac{2y_3}{c}\right) - \frac{qE^2 s_1}{(s_1 E + s_2 y_3)^2} - \sigma_2 u_3 & 0 & -\sigma_2 y_3 \\ \sigma_3 u_3 - \sigma_5 w_3 & \sigma_3 u_3 - \sigma_5 w_3 & -\sigma_4 - \sigma_5 y_3 & \sigma_3 y_3 \\ \sigma_5 w_3 - \sigma_3 u_3 & \sigma_5 w_3 - \sigma_3 u_3 & \sigma_5 y_3 & -\sigma_3 y_3 - \sigma_6 \end{bmatrix}$$

One of the Eigen values is $\lambda_{31} = r_1 - \alpha y_3 - \sigma_1 u_3$ if

All other Eigen values $\lambda_{32}, \lambda_{33}$and λ_{34} are roots for the following equation:

$$\lambda^3 + B_1\lambda^2 + B_2\lambda + B_3 = 0$$

$$B_1 = a_{11+}a_{12},\ \ B_2 = a_{11}a_{12} + a_{13} + a_{14}, B_3 = a_{12}a_{13} + \sigma_4 a_{14}$$

With $a_{11} = \sigma_4 + \sigma_6 + (\sigma_3 + \sigma_5)y_3 > 0$

$$a_{12} = \sigma_2 u_3 + \frac{qE^2}{(s_1E+s_2y_3)^2} + \frac{2r_2y_3}{c} - r_2$$

$$a_{13} = \sigma_4\sigma_6 + (\sigma_3\sigma_4 + \sigma_5\sigma_6)y_3 > 0$$

$$a_{14} = (\sigma_5 w_3 - \sigma_3 u_3)\sigma_2 y_3$$

$$B_1B_2 - B_3 = a_{12}(a_{11}^2 - a_{13}) + a_{14}(a_{12} + a_{11} - \sigma_4) + a_{11}a_{13} + a_{11}a_{12}^2 + a_{12}a_{14} + a_{12}a_{13}$$

Since $a_{11}^2 > a_{13}$ so, if the following condition holds

$a_{12} > 0$ and $a_{14} > 0$ (12)

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then, all Routh-Hurwitz criteria $B_1 > 0, B_3 > 0$ and $B_1B_2 - B_3 > 0$ are satisfied, and hence λ_2, λ_3 and λ_4 have negative real part. Consequently, in addition to condition 12, the following condition hold.

$$r_1 < \alpha y_3 + \sigma_1 u_3 \tag{13}$$

then the system is locally asymptotically stable around the prey free equilibrium point

Jacobian matrix of system2 at E_4 is given:

$$J(E_4) =$$

$$\begin{bmatrix} r_1\left(1-\frac{2x_4}{k}\right)-\frac{\alpha y_4}{(1+\alpha T x_4)^2}-\sigma_1 u_4 & -\frac{\alpha x_4}{1+\alpha T x_4} & 0 & -\sigma_1 x_4 \\ \frac{r_2 {y_4}^2}{(x_4+c)^2} & r_2\left(1-\frac{2y_4}{x_4+c}\right)-\frac{qE^2 s_1}{(s_1E+s_2y_4)^2}-\sigma_2 u_4 & 0 & -\sigma_2 y_4 \\ \sigma_3 u_4-\sigma_5 w_4 & \sigma_3 u_4-\sigma_5 w_4 & -\sigma_4-\sigma_5(x_4+y_4) & \sigma_3(x_4+y_4) \\ \sigma_5 w_4-\sigma_3 u_4 & \sigma_5 w_4-\sigma_3 u_4 & \sigma_5(x_4+y_4) & -\sigma_3(x_4+y_4)-\sigma_6 \end{bmatrix}$$

By Gershgorin theorem each eigen values lies within at least one of the following discs:

$$\left|\lambda - r_1\left(1-\frac{2x_4}{k}\right)+\frac{\alpha y_4}{(1+\alpha T x_4)^2}+\sigma_1 u_4\right| \le \frac{\alpha x_4}{1+\alpha T x_4}+\sigma_1 x_4$$

$$\left|\lambda - r_2\left(1-\frac{2y_4}{x_4+c}\right)+\frac{qE^2 s_1}{(s_1E+s_2y_4)^2}+\sigma_2 u_4\right| \le \frac{r_2 {y_4}^2}{(x_4+c)^2}+\sigma_2 y_4$$

$$|\lambda+\sigma_4+\sigma_5(x_4+y_4)| \le \sigma_3(x_4+y_4)+2|\sigma_3 u_4-\sigma_5 w_4|$$

$$|\lambda+\sigma_3(x_4+y_4)+\sigma_6| \le \sigma_5(x_4+y_4+2|\sigma_5 w_4-\sigma_3 u_4|$$

If the following condition hold, then all eigen values must be negative,

$$\left\{\begin{matrix} r_1\left(1-\frac{2x_4}{k}\right) < \frac{\alpha y_4}{(1+\alpha T x_4)^2}+\sigma_1 u_4 \\ r_2\left(1-\frac{2y_4}{x_4+c}\right) < \frac{qE^2 s_1}{(s_1E+s_2y_4)^2}+\sigma_2 u_4 \\ \left|r_1\left(1-\frac{2x_4}{k}\right)-\frac{\alpha y_4}{(1+\alpha T x_4)^2}-\sigma_1 u_4\right| > \frac{\alpha x_4}{1+\alpha T x_4}+\sigma_1 x_4 \\ \left|r_2\left(1-\frac{2y_4}{x_4+c}\right)-\frac{qE^2 s_1}{(s_1E+s_2y_4)^2}-\sigma_2 u_4\right| > \frac{r_2 {y_4}^2}{(x_4+c)^2}+\sigma_2 y_4 \\ \sigma_4 > (\sigma_3-\sigma_5)(x_4+y_4)+2|\sigma_3 u_4-\sigma_5 w_4| \\ \sigma_6 > (\sigma_5-\sigma_3)(x_4+y_4)+2|\sigma_3 u_4-\sigma_5 w_4| \end{matrix}\right\} \tag{14}$$

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Consequently, the interior equilibrium point is locally asymptotically stable, if it is existed and condition 13 holds.

Numerical Simulation

In this section, the effect of toxicant and predator harvesting numerically will be shown. the Catch ability coefficient to harvest predators have varied and rates at which prey and predator decreasing due to toxicant have varied to observe the dynamics of system 2. the values of the model parameters are selected as given in eq. 14-16 and solved system 2 numerically, this illustrated in in Fig .1-3, where curves with red, blue curve and black colors are x, y, w and u compartment, respectively

$r_1 = 2, r_2 = 1$, K=500, $T = 2, \alpha = 0.1, P = 50, c = 40 \;\; q = 0.1, \;E = 0.5, \;s_1 = s_2 = 1,$

$$\sigma_1 = 0.001, \;\sigma_2 = 0.001, \;\sigma_3 = 0.3, \;\sigma_4 = 0.1, \;\sigma_5 = 0.03 \; and \; \sigma_6 = 0.1 \qquad (15)$$

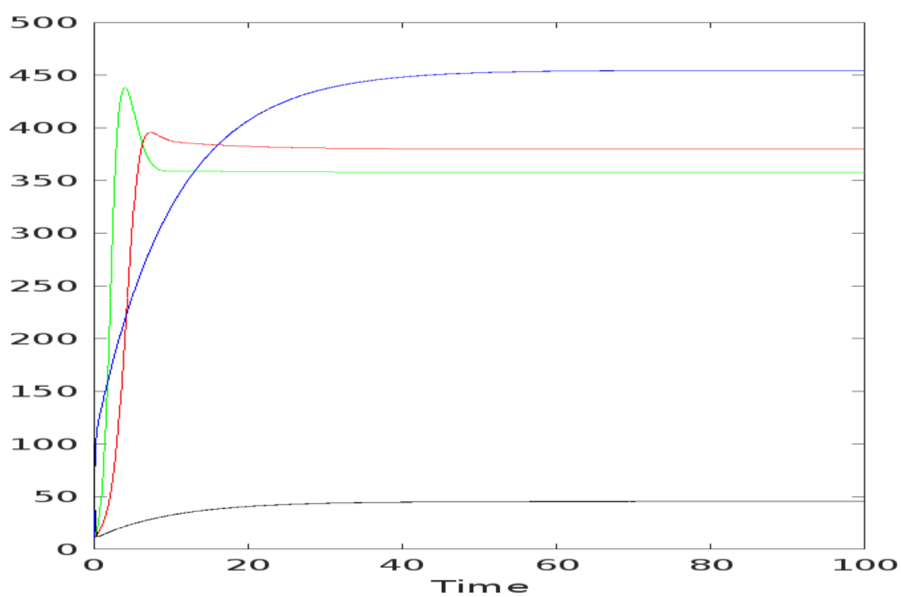

Figure 1. The time series is start at (10, 10, 10, 100) and the solution approaches asymptotically to the interior equilibrium point $(357.8285, 379.7012, 454.5524, 45.4299)$ For those parameter values, it is observed that system2persists and approaches asymptotically to the positive equilibrium point in the as shown in Figure 1.

Noting that parameter values given by eq. 15, satisfy the condition14. which confirms the analytical result. Now if the rates at which prey and predator decreasing due toxicant is increased, it will see that system2 approach prey free equilibrium point as shown in Figure2.

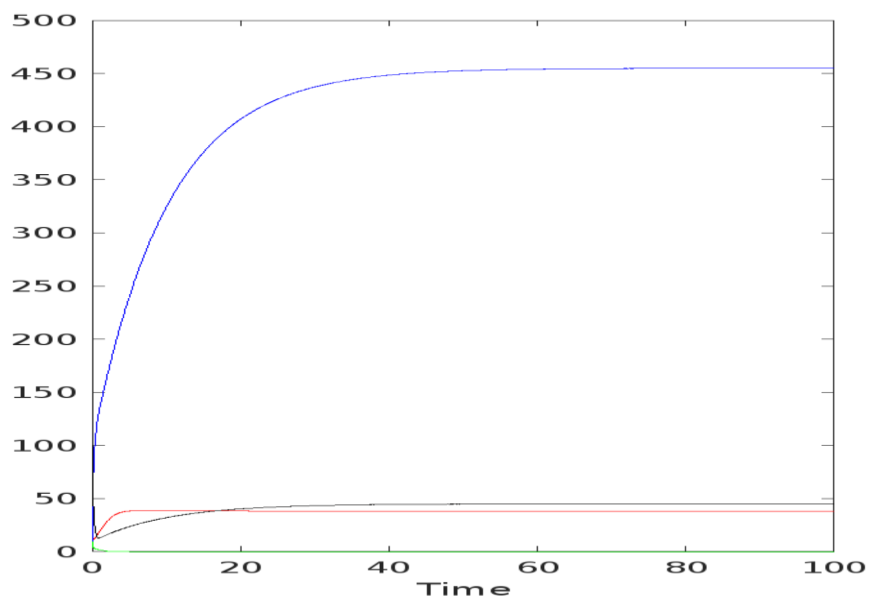

Figure 2. Time series started at (10, 10, 10, 100) solution of the system2 with parameter values in eq.5 and change value σ_1 to 0.1, approaches an asymptotically to free equilibrium point $(0, 38.1439, 454.8892, 45.0930)$. Note that when the value of σ_1 changed to 0.1, then parameter values satisfy conditions 12 and 13, therefore, it confirm that analytical results for stability state of prey free equilibrium point.

Now if the value of the parameters relative to, predator decreasing due toxicant and harvesting predator is increased and fixed other parameters, as given in eq.15, then solve system2 numerically as shown in Figure 3.

Noting that parameters using in above figure, satisfy the condition 10 and 11, meaning that the analytical result is correct. And note that Figure 3. show that both toxicant and harvesting have negative role predator individuals and may predator population will disappear due to existence of both toxicant and harvesting predator

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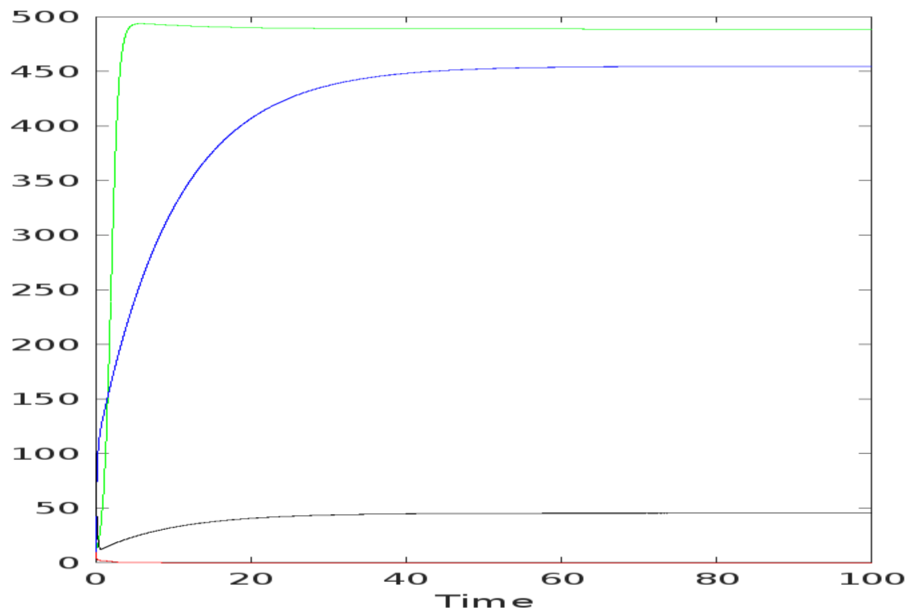

Figure 3. Time series started at (10, 10, 10, 100), solution of system 2 with parameter values eq.15 with change (σ_2 to 0.1 or q to 10)approaches predator free equilibrium point $(488.6438, 0, 454.5662, 45.4161)$.

Conclusion:

In this paper, Leslie Gower predator prey model is modified by taking in to account toxicant of environment and harvesting of predator individuals. The model has at most four-equilibrium point, population, predator free, and prey free and interior equilibrium points. It is shown that the population free equilibrium is not stable while other equilibrium will be stable under certain conditions. Effecting of both toxicant and predator harvesting is shown numerically, it has been seen that for a small change of values of parameters relative to toxicant and harvesting factor, dynamics of system2 changes.

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Conflict of interests.

There are non-conflicts of interest.

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