ARTICLE
JOURNAL OF UNIVERSITY OF BABYLON
For Pure and Applied Sciences (JUBPAS)
Vol.31; No.1.| 2023

Pn –Ideal of Commutative Ring

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Pn - مثالي في الحلقة التبادلية

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Received:	2/2/2023	Accepted:	27/3/2023	Published:	31 /3/2023

ABSTRACT

Background:

This study gives a new generalization to Ids called Pn-Id. If for all $a, b \in R$ with $ab \in I$ and $a \notin \sqrt{0}$ then $b \in \sqrt{I}$, and a proper Id P of R is known as a $Pn-$ Id. It investigates some properties for example every element in $\boldsymbol{I}$ is nilpotent if $\boldsymbol{I}$ is an $\boldsymbol{Pn}-$ Id of $\boldsymbol{R}$, of Pn-Ids analogous to n-Ids and PI. Some characterizations such as If I is a Pn −Id of R, then I is also J −Id for generalization and it is proved that every element in Pn- Ids is nilpotent. Accordingly, New versions of some theorems and proposition about Pn-Ids are given.

Materials and Methods:

In this paper we used the $n-$ ideal and $r-$ ideal to define Pn −ideal.

Results:

This strategy is continued in the second half of the study, when piecemeals are introduced as a generalization of $n-$ Id. A PI I of R is said to be a Pn −Id if the condition $ab \in I$ with $a \notin \sqrt{0}$ implies $b \in \sqrt{I}$. for all $a, b \in R$. The notion of $Pn-$ Id is given and some properties of $Pn-$ Ids are investigated like to n −Ids. In Lemma 2.2, obtain every n −Id is Pn −Id. Also, if I is a $Pn-I$ iff $\sqrt{I}$ is an $n-I$. It is proved (Proposition 2.5) that If $J/_I$ is a $pn-$ Id in $R/_I$, then J is a $Pn-$ Id in R.

Conclusion:

This study provides a new generalization to Ids called Pn-Id. If for all $a, b \in R$ with $ab \in I$ and $a \notin \sqrt{0}$ then $b \in \sqrt{I}$, then a proper Id P of R is known as a Pn −Id. Some properties of Pn-Ids analogous to n-Ids and PI are investigated. Giving characterizations for such generalization proved that every element in I is nilpotent, when I is a Pn-Id. Consequently, new versions of some theorems and proposition about Pn-Ids are given.

Key words:

PI, n-ideal, r-ideal and primary Id.

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الخلاصة

مقدمة:

في هذا البحث نعطي تعميما جديدا ل مثالي العليا تسمي PN – المثالي. اذا كان كل $a,b \in R$ مع $ab \in I$ و $a \notin \sqrt{0}$ ثم $b \in \sqrt{I}$, فان P المثالي المناسب ل R يعرف باسم Pn المثالي. وندرس بعض خصائص مثالي العليا Pn المماثلة ل I– n و PI. وكذالك نعطي بعض التوصيفات لمثل هذا التعميم ونثبت ان كل عنصر في Pnمثالي عديم القوة , عندما يكون مثاليا ل Pn. أخيرا, نقدم بعض النظريات و الاقتراحات جديدة عن Pn مثالي.

طرق العمل:

في هذا البحث استخدمنا n – الثالي و r – المثالي لتعريف Pn – المثالي

الاستنتاجات:

اعطيا تعريفا جديدا ل Pn المثالي وكذالك نعطي بعض التوصيفات لمثل هذا التعميم ونثبت ان كل عنصر في Pnمثالي عديم القوة, عندما يكون مثاليا ل Pn. أخيرا, نقدم بعض النظريات و الاقتراحات جديدة عن Pn مثالي. مثلاً , كل n مثالي يكون Pn مثالي تحت شرط.

الكلمات المفتاحية:

المثالي الأولي, n المثالية , r المثالية , مثالي ابتدئي

INTRODUCTION

R will be a commutative ring with nonzero identity for duration of this paper in [1], $\sqrt{I} = \{a \in R | a^n \in I \; for \; some \; n \in N\}$ is used to represent the radical I. It is specifically referred to as the set of all nilpotents in R as $\sqrt{0}$, that means $\{a \in R | a^n = 0 \; for \; some \; n \in N\}$ [1]. In [2], A PI P of R is a proper Id with the property that for any $a, b \in R$, $ab \in P$ implies either $a \in P$ or $b \in P$. In commutative ring theory, primary Ids play a significant role. A proper Id A of a ring R is primary if $pq \in A$ implies $p \in Q$ or $q \in \sqrt{A}$. In[3], A proper Id If whenever $a, b, c \in R$ and $abc \in I$, then $ab \in I$ or $c \in \sqrt{I}$, then I of R is defined to as a $1-$absorbing PI of R. In [4], A proper Id P of a, ring R is known as an $I-$ PI if when,ever $a, b \in R$ with $ab \in P - IP$ then $a \in$ P or b ∈ P. In [5], A proper Id If whenever $a, b, c \in R$ and $abc \in I$, then $ab \in I$ or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$, then I of R is referred to as a $2-$ absorbing primary Id of R. In [6], A proper Id when $a, b \in R$ with $ab \in I$ and $a \notin J(R)$, then $b \in I$ where $J(R)$ is the Jacobson radicall of R, then I of a ring R is referred to as a $J-$ Id.

Recently, In [7], Tekir, Koc and Oral the class of $n-$ Id was defined and researched as a subclass of $r-$Id. If whenever $a, b \in R$ with $ab \in I$ and $a \notin \sqrt{0}$ then $b \in I$, then a proper Id I of R to be an n-Id. In [8], Mohamadian if when,ever $a, b \in R$ with $ab \in I$ and if $ann(a) = 0$, it follows that $b \in I$, where $ann(a)$ =$\{r \in R \,| ar = 0\}$. In [9] Weakly primary Ids were first introduced and studied by Ebrahimi Atani and Farzali pour in 2005.In [10], A proper Id P of a commutative ring R *is* weakly primary if $0 \neq pq \in P$ implies $p \in P$ or $q \in \sqrt{P}$.

This strategy is continued in the second half of the study, when piecemeals are introduced as a generalization of $n-$ Id. A PI I of R is said to be a Pn $-$Id if the condition $ab \in I$ with $a \notin \sqrt{0}$ implies $b \in \sqrt{I}.$ for all $a, b \in R$. The notion of $Pn-$ Id is geven and some properties of $Pn-$ Ids are investigated like to $n-$Ids. In Lemma 2.2, obtain every $n-$I is $Pn-$Id. Also, if I is a $Pn-I$ iff $\sqrt{I}$ is an $n-I$. It is proved (Proposition 2.5) that If $J/_I$ is a $pn-$ Id in $R/_I$, then J is a $Pn-$ Id in R.

2. Pn –Ideal

The aim of this section is to study the Pn - ideals in commutative rings.

Definition 2.1. A proper Id $\boldsymbol{I}$ of $\boldsymbol{R}$ is said to as a $\boldsymbol{Pn}-$ Id if and only if $\boldsymbol{a}, \boldsymbol{b} \in \boldsymbol{R}$ with $\boldsymbol{ab} \in \boldsymbol{I}$ and $\boldsymbol{a} \notin \sqrt{\mathbf{0}}$ then $\boldsymbol{b} \in \sqrt{\boldsymbol{I}}.$

Lemma 2.2. Every $\boldsymbol{n}-$ Id is $\boldsymbol{Pn}-$ Id. However, the opposite is not generally true.
Consider the I $< 8 > = \{8n : n \in \mathbb{Z}\}$ in $\mathbb{Z}$, is Pn $-$Id, but not $n-$ Id, since $2{,}4 \in \mathbb{Z}, 2.4 = 8 \in < 8 >$ and $2^n \neq 0$ but $4 \notin < 8 >$.

Proposition 2.3. Every element in $\boldsymbol{I}$ is nilpotent if $\boldsymbol{I}$ is an $\boldsymbol{Pn}-$ Id of $\boldsymbol{R}$.
Proof: Let I be an Pn $-$Id of R. Assume that $I \nsubseteq \sqrt{0}$. Then there is $a \in I$ and $a \notin \sqrt{0}$, sin,ce $a.1 = a \in I$, and I is a $Pn-$ Id then $1 \in \sqrt{I}$ and $1^n \in I$ for some $n \in N$ imples that $1 \in I, then\ I = R$, a contradiction. Hence $I \subseteq \sqrt{0}.$

Proposition 2.4. Let $\boldsymbol{I}$ be proper Id of $\boldsymbol{R}$. Then $\boldsymbol{I}$ is a $\boldsymbol{Pn}-$ Id if and only if $\sqrt{\boldsymbol{I}}$ is an $\boldsymbol{n}-$ Id.
Proof: Consider that I is a Pn –Id. Assume $ab \in R, a, b \in \sqrt{I}$, $a \notin \sqrt{0}$. Since $a^n b^n \in I, a^n \notin \sqrt{0}$, $b^n \in \sqrt{I}$. $(b^n)^r$. Then $b \in \sqrt{I}$.

Conversely, let $\sqrt{I}$ be an $n-I$. Let $a, b \in R, ab \in I$ and $a^n \neq 0$, since $I \subseteq \sqrt{I}$, then $ab \in \sqrt{I}$ and $a^n \neq 0$. $\sqrt{I}$ is a n –I, therefore $b \in \sqrt{I}$.

Proposition 2.5. Let $\boldsymbol{I}$ be an $\boldsymbol{Pn}$ $-$Id contained in $\boldsymbol{J}$. If $\boldsymbol{J}/_{\boldsymbol{I}}$ is a $\boldsymbol{pn}-$ Id in $\boldsymbol{R}/_{\boldsymbol{I}}$, then $\boldsymbol{J}$ is a $\boldsymbol{Pn}-$ Id in $\boldsymbol{R}$.
Proof: Assume I , $J/_I$ be two $Pn-I$ds. To show J is $Pn-$ Id in R. Let $a, b \in R, ab \in J$ and $a^n \neq 0$. There are 2 cases:

Case 1: If $(a + I)^n = a^n + I = I$ then $a^n \in I, a \in \sqrt{I}$, since $I \subseteq J$ then $\sqrt{I} \subseteq \sqrt{J}$ and we get $a \in \sqrt{J}$.

Case 2: If $(a + I)^n = a^n + I \neq I, then\ (a + I)(b + I) = (ab) + I \in \sqrt{J/_I}$. so$(b + I)^n = b^n + I \in J/_I$ imples $b^n \in J$ and $b \in \sqrt{J}$. Hence J is a $Pn - I$d in R.

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Proposition 2.6. Let $\{\boldsymbol{I_i}\}_{\boldsymbol{i\in N}}$ be a non – empty set of $\boldsymbol{Pn}-$Ids of $\boldsymbol{R}$. Then, $\cap_{\boldsymbol{i\in N}}\boldsymbol{I_i}$ is a $\boldsymbol{Pn}-$Id of $\boldsymbol{R}$.

Proof: Assume $ab\in\cap_{i\in N}I_i$ with $a\notin\sqrt{0}$, $\forall\, a,b\in R$. Then $ab\in I_i$ for all $i\in N$. Since I_i is a $pn-$ Id , then $b\in\sqrt{I_i}$ and so $b\in\cap_{i\in N}\sqrt{I_i}=\sqrt{\cap_{i\in N}I_i}$.

Proposition 2.7. A PI $\boldsymbol{P}$ of $\boldsymbol{R}$, is $\boldsymbol{Pn}-$ Id iff $\boldsymbol{I}=\sqrt{\boldsymbol{0}}$.

Proof: Assume that I is a PI of R. It is obvious that $\sqrt{0}\subseteq I$. If it is $Pn-$ Id of R, by Propo,sition 2.3 $I=\sqrt{0}$ since $I\subseteq\sqrt{0}$.

For the opposite, suppose $I=\sqrt{0}$. Let $ab\in I, a\notin\sqrt{0}, for\ a,b\in R$. then $b\in I$ because I is prime and $a\notin\sqrt{0}$, I is Pn-Id of R because since $I=\sqrt{I}$.

Corollary 2.8. $\sqrt{\boldsymbol{0}}$ is an $\boldsymbol{Pn}-$ Id of $\boldsymbol{R}$ iff it is a PI of $\boldsymbol{R}$.

Proof: Let $ab\in\sqrt{0}$ and $a\notin\sqrt{0}$. Since $\sqrt{0}$ is an $Pn-$ Id of R, we conclude that $b\in\sqrt{0}$. Hence $\sqrt{0}$ is a PI of R, conversely Let$\sqrt{0}$ is a PI of R,then by Proposition 2.8 [10], if $\sqrt{0}$ is a $n-$ Id of R and by Lemma 2.2 , $\sqrt{0}$ is a $Pn-$ Id of R

Proposition 2.9. For any ring $\boldsymbol{R}$, if every proper Id is $\boldsymbol{pn}-$Id, then $\boldsymbol{R}$ has a uniqe PI which is $\sqrt{\boldsymbol{0}}$.

Proof: Let P be a PI of R then by Proposition 2.8 [10], we obtain $P=\sqrt{0}$ when necessary.Additionally, $\sqrt{0}$ is a maximalI of R.

Proposition 2.10. For any ring R, if every proper principle Id is $\boldsymbol{n}-$ Id, then every proper Id is a $\boldsymbol{Pn}-$ Id.

Proof: Suppose I be a pro,per Id of R and $ab\in I$, where $a\notin\sqrt{0}$. Since $ab\in\langle ab\rangle$ and $\langle ab\rangle$ is $n-$Id of R, we conclude that $b\in\langle ab\rangle\subseteq I\subseteq\sqrt{I}$. Then, I is a $Pn-$ Id of R.

Theorem 2.11. Let S be a multiplicatively closed subset of $\boldsymbol{R}$ and let I be an Id of R such that $\boldsymbol{I}\cap\boldsymbol{S}=\emptyset$. Then, a Pn-Id J belong in I such that $\boldsymbol{J}\cap\boldsymbol{S}=\emptyset$.

Proof : Consider the set $\beta=\{I':I'$ is anI of R with $I'\cap S=\emptyset\}$ Sincee $I\in\beta$, there are $\beta\neq\emptyset$. By Zorn's lemma [7], J becomes our maximum member. we must prove that J is an $n-$Id of R. Assume that not. if $a,b\in R\ such\ that\ ab\in J, a\notin\sqrt{0}$ and $b\notin J$. Thus $b\in(J:a)$ and $J\subseteq(J:a)$. By the maximality of J, we know $(J:a)\cap S\neq\emptyset$, and thus there exists an $s\in S$ and S is an multiplicatively closed subset of R. Thus, $S\cap J\neq\emptyset$ and this contradicts with $J\in\beta$ Hence J is a $Pn-$Id of R.

Proposition 2.12. If I is a $Pn-$Id of R, then I is also $J-$Id.

Proposition 2.13. If $\sqrt{0}$ is $Pn-$Id of R, then it is a primary Id of R.

Corollary 2.14. If $\sqrt{\boldsymbol{0}}$ is $\boldsymbol{Pn}-$Id of R, then it is a $\boldsymbol{2}-$ absorbing primary Id of $\boldsymbol{R}$.

Theorem 2.15. If $\sqrt{\boldsymbol{0}}$ is $\boldsymbol{1}-$ absorig primary Id of $\boldsymbol{R}$ if and only if $\sqrt{\boldsymbol{0}}$ is $\boldsymbol{Pn}-$Id of $\boldsymbol{R}$.

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Proof: Let $ab \in \sqrt{0}$ for $a, b \in R$, it may be assumed that a, b are nonunits element of R. Let $n \geq 2$ be an even positive integer such,that $(ab)^n \in \sqrt{0}$. Then $n = 2m$ for some positive integer $m \geq 1$. Since $(ab)^n = a^n b^n = a^m a^m b^n \in \sqrt{0}$ and $\sqrt{0}$ is a $1 -$ absobrnig PI. It is concluded, that $a^m a^m = a^n \in \sqrt{0}$ or $b^n \in \sqrt{0}$. So $a \in \sqrt{0}$ or $b \in \sqrt{0}$. $\sqrt{0}$ is a PI so $\sqrt{0}$ is Pn $-$Id of R.

Corollary 2.16.

1. If$\sqrt{0}$ is Pn $-$Id of R, then it is weakly primary Id of R.
2. If $\sqrt{\mathrm{I}}$ is pn $-$Id of R, then it is a primary Id of R.
3. If $\sqrt{\mathrm{I}}$ is pn $-$Id of R, then it is $1 -$ absorbing primary Id of R.
4. If $\sqrt{\mathrm{I}}$ is pn $-$Id of R, then it is $2 -$absorbing primary Id of R.

Theorem 2 .17. If $\sqrt{\mathbf{0}}$ is $\boldsymbol{I} -$ prime and $\left(\sqrt{\mathbf{0}}\right)^{2} \not\subseteq \boldsymbol{I}\sqrt{\mathbf{0}}$, then $\sqrt{\mathbf{0}}$ is an $\boldsymbol{Pn} -$ Id.

Proof: By Theorem 2.2 [6], $\sqrt{0}$ is a PI, then by Corollary 2.8, $\sqrt{0}$ is n $-$Id, since $\sqrt{\sqrt{0}} = \sqrt{0}$. The result is $\sqrt{0}$ is pn $-$Id.

Conflict of interests.

There are non-conflicts of interest.

References

[1] M. F. Atiyah and I. G. MacDonald. Addison-Wesley, Reading, MA: Introduction to Commutative Algebra, 1969.
[2] H. Matsumura. Published Continuousely : Commutative Ring Theory, 1584.
[3] A. Badawi and E. Yetkin, "On 1-Absorbing primaryIs of commutative rings", Journal of Algebra and Its Application, vol.19, no.06, May 2019.
[4] I. Akray, "I – Prime Ideal ", Journal of Algebra and Related Topics vol. 4, no 2, pp. 41-47, 2016.
[5] A. Badawi, U. Tekir and E. Yetkin, "On 2-absorbing primaryIs in commutative rings", Bull. Korean Math. Soc., Vol.51, no.4 , pp.1163-1173,2014.
[6] H. A. Khashan and A. B. Bani, "J-Ideals of Commutative Rings ", Journal of International Electronic Algebra, Vol. 29, pp.148-1642021.
[7] U. Tekir, S.Koc and K. H.Oral, " n-Ideals of commutative rings", Journal of
Filomat, vol.31, no.(10), pp.2933-2941, Jan.2017.
[8] R. Mohamadian, " r-ideals in commutative rings", Journal of Turk Math , Vol.39, pp.733-749, 2015.
[9] S. E. Atani and F. Farzalipour,"On weakly primaryIs", Journal of Georgian Mathematical, vol.12, no.3, pp.423,2005.
[10] A. Badawi, U. Tekir and E. Yetkin, "On weakly 2-absorbing primaryIs of commutative rings", Jornal of Korean Math. Soc, Vol.52, no.(1), pp.97-111,2015.

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