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Some Connections About Rad_g-lifting Modules

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بعض العلاقات حول مقاسات الرفع من النمط-Rad_g

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ABSTRACT

In this work, we discussed certain relationships between the idea of Rad_g-lifting modules and many other related types of modules, explained these relationships and provided criteria that make these modules comparable to Rad_g-lifting modules.

CONCLUSION

It has addressed to the concept of Rad_g-lifting and other related types of modules and analyze the relations between these different kinds of modules like g-local, ⊕-g-supplemented, g-supplemented, sgrs-modules, $sgrs^{\oplus}$-module, $(P_g{}^*)$-modules etc. The necessary conditions have been set to achieve equivalence between them and thus obtained equivalent definitions to the Rad_g-lifting definition within certain conditions.

KEYWORDS

g-small submodules, g-hollow, g-lifting, sgrs-modules, $sgrs^{\oplus}$-modules, Rad_g-lifting.

الخلاصة

ناقشنا في هذا العمل علاقات معينة بين فكرة مقاسات الرفع من النمط –Rad_gوالعديد من أنواع المقاسات الأخرى ذات العلاقة، وضحنا تلك العلاقات، وقدمنا معايير تجعل هذه المقاسات قابلة للمقارنة مع مقاسات الرفع من النمط –Rad_g.

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1. INTRODUCTION

The modules in this paper are all unitary left R-modules, where R is an associative ring with unity. β is called large in μ, if $\eta \cap \beta \neq 0$ for any nonzero submodule η of μ, written as $\beta \trianglelefteq \mu$ [13]. A submodule $\beta \neq \mu$ called small if, $\mu = \beta + \eta$ then $\eta = \mu$ for $\eta \leq \mu$. If $\beta \leq \mu$ then β is said to be *e*-small if, $\mu = \beta + \eta$ then $\eta = \mu$ for any large submodule η of μ. Some other authors recalled the term "*e*-small" as a "g-small", denoting $\ll_g$ which we will use in this paper. It is obvious that every "small" submodule is "g-small". A generalization of radical for a module μ was introduced by Zhou and Zhang [1], which denote by $Rad_g(\mu)$, where they defined it as; $Rad_g(M) = \sum\{\beta | \beta \ll_g \mu\} = \cap\{\beta \trianglelefteq \mu | \beta \text{ } is\ maximal\ in\ M\}$. In [14] introduced the definition of the uniform module as a nonzero module in which all its nonzero submodules are large. A module which has a unique submodule which is maximal is called local. If a module that has all submodules small, then this module is called hollow. Hadi and Aidi [2] studied a generalized hollow module by given its definition as a module which all its submodules are g-small. Clearly, every local module is hollow and hence generalized hollow. Recall from [3] that a submodule η has g-supplement submodule $\mathcal{B}$ in μ if and only if $\eta + \mathcal{B} = \mu$ and $\eta \cap \mathcal{B} \ll_g \mathcal{B}$. A module μ is called g-supplemented if all its submodules have g-supplement in μ. We know [4] that a module μ is ⊕-g-supplemented if, every direct summand submodule of μ has a g-supplement. Furthermore, in [5] gave the concept of g-lifting module as, for each $\eta \leq \mu$, there is a decomposition $\mu = \gamma \oplus \beta$ where $\gamma \leq \eta$ with $\eta \cap \beta \ll_g \mu$. Later [6] define a g-local module as a module which has a g-small and maximal generalized radical. Ghawi [7] presented the definition of g-cover of a module μ as a pair (ρ, f) where ρ is a module and $f: \rho \to \mu$ is an epimorphism with $Kerf \ll_g \rho$. If ρ is a projective module, then the pair (ρ, f) a projective g-cover of μ. A module μ is Rad_g-lifting if, for any $\eta \leq \mu$ with $Rad_g(\mu) \subseteq \eta$, there is $\mu = \varrho \oplus \beta$ with $\varrho \leq \eta$ and $\eta \cap \beta \ll_g \beta$. This definition was introduced by Mirza and Ghawi [8] as a proper generalization of g-lifting modules. The study's main objective is to highlight and analyze several results that explain the connections between the concept of Rad_g-lifting and other classes of modules as well as; g-local, ⊕-g-supplemented, g-supplemented, sgrs-modules, $sgrs^{\oplus}$-modules, $(P_g{}^*)$-module etc.

2. $\mathrm{Rad_g}$-LIFTING MODULES WITH CONNECTIONS

We will start with some results that show the relationship between $\mathrm{Rad_g}$-lifting module and some related concepts.

Proposition 2.1. Let μ be a module such that $Rad_g(\mu) \neq \eta$ for any $\eta \leq \mu$. If $Rad_g(\mu) \leq \mu$ which, is maximal, then μ is a $\mathrm{Rad_g}$-lifting module.

Proof. Let $\eta \leq \mu$ where $Rad_g(\mu) \subseteq \eta$. By the assumption, $Rad_g(\mu) \subset \eta$. Since $Rad_g(\mu)$ is maximal of μ, so $\eta = \mu$, a trivially, $\mu = \mu \oplus (0)$ where $\mu \leq \eta$ and $\eta \cap (0) \ll_g (0)$. Hence, μ is a $\mathrm{Rad_g}$-lifting module. □

Proposition 2.2. A g-local module is $\mathrm{Rad_g}$-lifting.

Proof. Suppose μ is a g-local module. Thus, we have $Rad_g(\mu)$ is a maximal and g-small submodule of μ. Let $\eta \leq \mu$ with $Rad_g(\mu) \subseteq \eta$. If $Rad_g(\mu) = \eta$, then η is g-small in μ, trivially, $\mu = (0) \oplus \mu$ such that $(0) \leq \eta$ and $\eta \cap \mu = \eta \ll_g \mu$. Now let $Rad_g(\mu) \subset \eta$. As $Rad_g(\mu)$ is maximal in μ, thus $\eta = \mu$, a trivially, $\mu = \mu \oplus (0)$ such that $\mu \leq \eta$ and $\eta \cap (0) \ll_g (0)$. Thus, μ is $\mathrm{Rad_g}$-lifting. □

The opposite direction of Proposition 2.2 incorrect, in general, as seen below.

Example 2.3. Assume $R = \mathbb{Z}$ and $\mu = \mathbb{Q}$. We have $\mathbb{Q}$ as $\mathbb{Z}$-module is $\mathrm{Rad_g}$-lifting, see [8, Remark 2.7]. But $Rad_g(\mathbb{Q})$ dose not a maximal submodule of $\mathbb{Q}$ (in fact, $Rad_g(\mathbb{Q}) = \mathbb{Q}$), and hence, not g-local.

We Know from [9] a module μ is called sgrs-module if every $\gamma \leq \mu$ including, $Rad_g(\mu)$, has a g-supplement inside μ. Later Kamel and Ghawi [10] introduced $\mathrm{sgrs}^{\oplus}$-module which they have defined as, if for any $\varrho \leq \mu$ with $Rad_g(\mu) \subseteq \eta$, there is $\varrho \leq^{\oplus} \mu$ such that $\mu = \eta + \varrho$ and $\eta \cap \varrho \ll_g \varrho$.

Proposition 2.4. Every $\mathrm{Rad_g}$-lifting module is a $\mathrm{sgrs}^{\oplus}$-module, and hence it is a sgrs-module.

Proof. Suppose μ is a $\mathrm{Rad_g}$-lifting module, $\eta \leq \mu$ such that $Rad_g(\mu) \subseteq \eta$. Thus there is $\mu = \varrho \oplus \beta$ where $\varrho \leq \eta$ and $\eta \cap \beta \ll_g \beta$. Therefore, $\mu = \eta + \beta$ where $\beta \leq^{\oplus} \mu$ with $\eta \cap \beta \ll_g \beta$. Hence μ is a $\mathrm{sgrs}^{\oplus}$-module. □

As an application of the above consequence, the $\mathbb{Z}$ does not be a $\mathrm{sgrs}^{\oplus}$-module as a $\mathbb{Z}$-module, hence it is not $\mathrm{Rad_g}$-lifting, see [10, Examples 2.7(3)].

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Proposition 2.5. Let μ be an indecomposable module. If μ is a sgrs$^{\oplus}$-module, then μ is Rad_g-lifting.

Proof. Let μ be a sgrs$^{\oplus}$-module, $\eta \leq \mu$ and $Rad_g(\mu) \subseteq \eta$. If $\eta = \mu$, trivially, there is a decomposition $\mu = \mu \oplus (0)$ where $\mu \leq \eta$ and $\eta \cap (0) \ll_g (0)$. Suppose $\eta \neq \mu$. Then there exists a direct summand β of μ where $\mu = \varrho \oplus \beta = \eta + \beta$ with $\eta \cap \beta \ll_g \beta$. Since μ is indecomposable, it means $\varrho = 0$. Therefore $\varrho \leq \eta$, and hence μ is a Rad_g-lifting module. □

The following consequence is immediately from Propositions 2.4 and 2.5.

Corollary 2.6. Let μ be an indecomposable module. Then μ is a sgrs$^{\oplus}$-module if and only if μ is a Rad_g-lifting module.

From [5] we know that every semisimple module is g-lifting, and consequently, it is Rad_g-lifting module. Ghawi in [12] defined $(P_g{}^*)$-module as a module μ which for $\eta \leq \mu$, there exists $\beta \ll^{\oplus} \mu$ and $\beta \leq \eta$ with $\eta/\beta \subseteq Rad_g(\mu/\beta)$. There is no direct relation between Rad_g-lifting module and $(P_g{}^*)$-module. The next proposition will give the case for these concepts to be equivalents.

Proposition 2.7. Let μ be a module, we have the assertions below:

(1) μ is semisimple.

(2) μ is g-lifting.

(3) μ is Rad_g-lifting.

(4) μ is a $(P_g{}^*)$-module.

Then $(1) \Longrightarrow (2) \Longrightarrow (3)$.

If $Rad_g(\mu) = 0$, then $(3) \Longrightarrow (4) \Longrightarrow (1)$.

Proof. $(1) \Longrightarrow (2) \Longrightarrow (3)$ Clear.

$(3) \Longrightarrow (4)$ Assume $\eta \leq \mu$. As $Rad_g(\mu) = 0 \subseteq \eta$, by (3) there is $\mu = \varrho \oplus \beta$ where $\gamma \leq \eta$ and $\eta \cap \beta \ll_g \beta$, so in μ, that implies $\eta \cap \beta \subseteq Rad_g(\mu)$. Hence μ is a $(P_g{}^*)$-module, by [4, Proposition 2.6].

$(4) \Longrightarrow (1)$ Let $\eta \leq \mu$ with $Rad_g(\mu) = 0$. By (4), there is $\mu = \beta \oplus \varrho$ such that $\beta \leq \eta$ and $\eta \cap \varrho \subseteq Rad_g(\mu)$. It follows that $\mu = \eta + \varrho$ and $\eta \cap \varrho = 0$. Thus, $\mu = \eta \oplus \varrho$, and (1) holds.□

As a direct confirmation of a previous example by invoking the above result, we know that the $\mathbb{Z}$-module $\mathbb{Z}$ does not be Rad_g-lifting, actually, $\mathbb{Z}$-module $\mathbb{Z}$ neither semisimple nor g-lifting, and $Rad_g(\mathbb{Z}) = 0$.

Corollary 2.8. Let μ be a module and η a nonzero Rad_g-lifting submodule of μ with $\eta \cap Rad_g(\mu) = 0$, then η is semisimple.

Proof. As $Rad_g(\eta) \subseteq \eta \cap Rad_g(\mu)$, then $Rad_g(\eta) = 0$ and hence η is semisimple by Proposition 2.7. □

The following corollary comes from corollary 2.8.

Corollary 2.9. Let μ be a module with $Rad_g(\mu) = 0$. If η is a nonzero Rad_g-lifting submodule of μ, then η is semisimple.

The following results give the necessary conditions for Rad_g-lifting module to be Artinian.

Corollary 2.10. If μ is a finitely generated Rad_g-lifting module with $Rad_g(\mu) = 0$. Then μ is an Artinian module.

Proof. From Proposition 2.7, μ is semisimple. Hence by[15, 31.3] μ is an Artinian. □

Proposition 2.11. Let μ be a finitely generated module that hold DCC on g-small submodules. If μ is a Rad_g-lifting module, then μ is Artinian.

Proof. Assume μ is a Rad_g-lifting module. By [8, Proposition 2.35], $\mu/Rad_g(\mu)$ is semisimple then $\mu/Rad_g(\mu)$ is finitely generated, and then $\mu/Rad_g(\mu)$ is Artinian by [15, 31.3]. Then, μ holds DCC on g-small submodules means $Rad_g(\mu)$ is Artinian, [5, Theorem 4]. Therefore, by [14, Theorem 6.1.2], μ is Artinian. □

Corollary 2.12. Let μ be a cyclic module that holds DCC on g-small submodules. If μ is a Rad_g-lifting module, then μ is Artinian.

Previously, we introduced results that declare the relationship between Rad_g-lifting module, $\text{sgrs}^{\oplus}$-module and sgrs-module. Now gives the necessary terms for equivalency.

Theorem 2.13. For a projective module μ. The assertions below are equivalent.

(1) For $\eta \leq \mu$ with $Rad_g(\mu) \subseteq \eta$, the factor μ/η has a projective g-cover

(2) μ is Rad_g-lifting.

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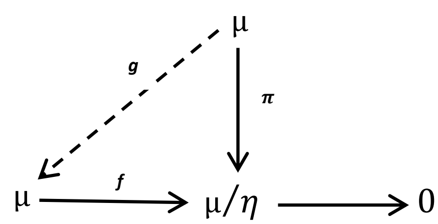
(3) μ is a sgrs$^{\oplus}$-module.

(4) μ is a sgrs-module.

Proof. (1) $\Longrightarrow$ (2) Suppose that $\eta \leq$ μ with $Rad_g(\mu) \subseteq \eta$. By (1), for a projective ρ and an epimorphism $\alpha: \rho \to \mu/\eta$ where $Ker\alpha \ll_g \rho$. let $\pi: \mu \to \mu/\eta$ be a canonical map, then there is a homomorphism $\beta: \mu \to \rho$ where $\alpha\beta = \pi$. Thus $\rho = \beta(\mu) + Ker\alpha$. Since $ker\alpha \ll_g \rho$, [1, Proposition 2.3] implies $\rho = \beta(\mu) \oplus \varrho$ where $\varrho \leq ker\alpha$ and ϱ semisimple, hence $\beta(\mu)$ is projective. So $\mu = Ker\beta \oplus \zeta$ for $\zeta \leq \mu$. It is easy to confirm that $Ker\beta \leq Ker\pi = \eta$. To prove that $\eta \cap \zeta \ll_g \zeta$. Since $Ker\alpha \ll_g \rho$, $Ker\alpha \cap \beta(\zeta) = \beta(\eta \cap \zeta) \ll_g \rho$. By [11, Lemma 2.12(1)], we deduce $\beta(\eta \cap \zeta)$ is g-small in $\beta(\mu) = \beta(\zeta)$. Hence $\eta \cap \zeta \ll_g \zeta$, as β is an isomorphism from ζ into $\beta(\zeta)$. Therefore μ is a $\mathrm{Rad_g}$-lifting module.

(2) $\Longrightarrow$ (3) $\Longrightarrow$ (4) By Proposition 2.4.

(4) $\Longrightarrow$ (1) Let $\eta \leq \mu$ with $Rad_g(\mu) \subseteq \eta$. By (4), there is $\varrho \leq \mu$ with $\mu = \eta + \varrho$ and $\eta \cap \varrho \ll_g \varrho$. Let $f: \mu \to \mu/\eta$ be a homomorphism defined by $f(x) = a + \eta$, where $x = n + a \in \mu$, $n \in \eta$ and $a \in \varrho$. Assume that $\pi: \mu \to \mu/\eta$ is a canonical map. Since μ is a projective, hence a homomorphism $g: \mu \to \mu$ and $fg = \pi$.



Then $f(g(\mu)) = \pi(\mu)$, so that $f^{-1}(f(g(\mu))) = f^{-1}(\mu/\eta)$, it follows that $\mu = g(\mu) + Kerf = g(\mu) + \eta \cap \varrho$. Since $\eta \cap \varrho \ll_g \varrho$, hence $\eta \cap \varrho \ll_g \mu$, by [1, Proposition 2.3], there is $\gamma \leq \eta \cap \varrho$ a where γ is semisimple with $\mu = g(\mu) \oplus \gamma$ and so that $g(\mu)$ is a projective R-module. Hence $g(\mu) \cong \mu/Kerg$ implies $Kerg \leq^{\oplus} g(\mu)$, so in μ, that means $\mu = Kerg \oplus \beta$ for some submodule β of μ. It follows that β is projective. Let $(fg)|_B$ Denote the restriction of fg on β, that is $(fg)|_B: \beta \to \mu/\eta$. Therefore $Ker((fg)|_\beta) \leq \eta \cap \varrho$. It follows that $Ker((fg)|_\beta) \ll_g \mu$. Since $Ker((fg)|_\beta) \subseteq \beta$ and $\beta \ll_g \mu$, [11, Lemma 2.12(1)] implies $Ker((fg)|_\beta) \ll_g \beta$. Hence β is a projective g-cover of μ/η. This completes the proof. □

Corollary 2.14 For a ring R, the assertions below are equivalent.

(1) For $I \leq R$ with $Rad_g(R) \subseteq I$, the factor R/I Has a projective g-cover.

(2) R is a $\mathrm{Rad_g}$-lifting ring.

(3) R is a sgrs$^{\oplus}$-ring.

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(4) R is a sgrs-ring.

Corollary 2.15. Let R be any semisimple ring, and μ be an R-module. Then an assertions below are equivalents.

(1) For $\eta \leq \mu$ with $Rad_g(\mu) \subseteq \eta$, the factor μ/η has a projective g-cover

(2) μ is a Rad_g-lifting ring.

(3) μ is a $\mathrm{sgrs}^{\oplus}$-ring.

(4) μ is a sgrs-ring.

Proof. As R is a semisimple ring, so by [14, Corollary 8.2.2(2)], an R-module μ is projective. Then the result is obtained by Theorem 2.13. □

The following proposition gives the equivalent assertion of Rad_g-lifting module under projectivety case.

Proposition 2.16. Let ρ be projective with $Rad_g(\rho) \ll_g \rho$. Then an assertion below are equivalent.

(1) ρ is a Rad_g-lifting.

(2) $\rho/Rad_g(\rho)$ is semisimple, and for every submodule η of ρ has $Rad_g(\rho)$ and $\bar{\eta} = \eta/Rad_g(\rho)$, there exists β direct summand of ρ with $\bar{\eta} = \bar{\beta}$.

Proof. (1) $\Longrightarrow$ (2) Let $\eta \leq \rho$ such that $Rad_g(\rho) \subseteq \eta$. Since ρ is a Rad_g-lifting module, thus by [8, Proposition 2.35], $\rho/Rad_g(\rho)$ Is semisimple. Put $\bar{\eta} = \eta/Rad_g(\rho)$. Again by [8, Proposition 2.2(3)] there is $\eta = \beta \oplus \varrho$ where $\beta \leq^{\oplus} \rho$ and $\varrho \ll_g \rho$. So $\varrho \subseteq Rad_g(\rho)$. It follows that $\beta + Rad_g(\rho) \subseteq \eta$. On the other hand, $\eta = \beta + \varrho \subseteq \beta + Rad_g(\rho)$, thus $\eta = \beta + Rad_g(\rho)$. Therefore, $\eta/Rad_g(\rho) = (\beta + Rad_g(\rho))/Rad_g(\rho)$, and hence $\bar{\eta} = \bar{\beta}$.

(2) $\Longrightarrow$ (1) Let $\eta \leq \rho$ such that $Rad_g(\rho) \subseteq \eta$. As $\rho/Rad_g(\rho)$ Is semisimple. We have that $\rho/Rad_g(\rho) = \eta/Rad_g(\rho) \oplus \varrho/Rad_g(\rho)$ for some submodule ϱ of ρ. By (2), there is $\gamma \leq^{\oplus} \rho$ with $\rho = \gamma \oplus \beta$ for some submodule β of ρ, and $\bar{\eta} = \bar{\gamma}$, so that $\varrho = \beta + Rad_g(\rho)$. It follows that $\rho = \gamma + \beta + Rad_g(\rho) = \eta + \beta$. As we know, $\rho = \eta + \beta$ is projective, [15, 41.14] implies $\rho = \eta' \oplus \beta$ with $\eta' \subseteq \eta$. Also, $\eta \cap \beta \leq \eta \cap \varrho = Rad_g(\rho) \ll_g \rho$, therefore ρ is a Rad_g-lifting module. □

According to [10], the definition of $P_g(\mu)$ as a module of all generalized radical submodules of μ.

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Proposition 2.17. Let μ be a Rad_g-lifting module such that $Rad_g(\mu) \neq \mu$. Then there is a decomposition $\mu = \gamma \oplus \varrho$ where ϱ is a g-supplement of $Rad_g(\mu)$ in μ, $Rad_g(\varrho) \ll_g \varrho$ and γ is a g-radical. In particular, if $P_g(\mu) = 0$, then $Rad_g(\mu) \ll_g \mu$.

Proof. Let $Rad_g(\mu) \neq \mu$. Since μ is a Rad_g-lifting module and $Rad_g(\mu) \subseteq Rad_g(\mu)$, by [5, Proposition 2.2(8)], there is $\gamma \leq \mu$ in $Rad_g(\mu)$ with $\mu = \gamma \oplus \varrho$ and ϱ a g-supplement of $Rad_g(\mu)$ in μ, i.e., $\mu = Rad_g(\mu) + \varrho$ and $Rad_g(\mu) \cap \varrho \ll_g \varrho$. Since $\varrho \leq^{\oplus} \mu$, then by [7, Lemma 2.12] $Rad_g(\mu) \cap \varrho = Rad_g(\varrho) \ll_g \varrho$. Also, from [6, Corollary 2.3], we get $\mu = Rad_g(\gamma) \oplus \varrho$. By the modular law, we get $\gamma = \gamma \cap (\varrho \oplus Rad_g(\gamma)) = Rad_g(\gamma) \oplus (\varrho \cap \gamma) = Rad_g(\gamma)$. Hence, ϱ is a g-radical. Now, if $P_g(\mu) = 0$, then $\gamma = 0$, which shows that $\mu = \varrho$, hence $Rad_g(\mu) \ll_g \mu$. □

The reverse of Proposition 2.17 need not be correct; in general, for instance, assume $\mu = R = \mathbb{Z}$, then $Rad_g(\mu) = 0 \neq \mu$. We have the decomposition $\mathbb{Z} = \mathbb{Z} \oplus 0$ where $\mathbb{Z}$ is a g-supplement of $Rad_g(\mu) = 0$, $Rad_g(\mathbb{Z}) = 0 \ll_g \mathbb{Z}$, with (0) is g-radical, while $\mu = \mathbb{Z}$ is not Rad_g-lifting $\mathbb{Z}$-module.

Next, some interesting results according to indecomposable condition.

Proposition 2.18. Let μ be an indecomposable module with $Rad_g(\mu) \neq \mu$. If μ is a Rad_g-lifting module, then $Rad_g(\mu) \ll_g \mu$. Moreover, if $Rad_g(\mu)$ is a maximal in μ, then the reverse is true.

Proof. Assume that μ is indecomposable and Rad_g-lifting module. Since $Rad_g(\mu) \subseteq Rad_g(\mu)$, so by Proposition 2.17, there is a unique decomposition $\mu = \mu \oplus 0$ such that μ is a g-supplement of $Rad_g(\mu)$ and 0 is a g-radical. Hence, $Rad_g(\mu) = Rad_g(\mu) \cap \mu$ is g-small in μ, as required. To prove the reverse assertion, let $Rad_g(\mu)$ be a maximal submodule of μ and $Rad_g(\mu) \ll_g \mu$, that is μ is g-local. Since μ is an indecomposable module, **[**6, Proposition 2.8**]** implies μ is local. By [2, Proposition 2.6] μ is generalized hollow, and thus by [8, Proposition 2.13], μ is a Rad_g-lifting module. □

The opposite direction of Proposition 2.18 need not be correct; in general, for instance, we know that $\mathbb{Z}$ is not Rad_g-lifting as a $\mathbb{Z}$-module, during the module $\mathbb{Z}_{\mathbb{Z}}$ is an indecomposable, $\mathbb{Z} \neq Rad_g(\mathbb{Z})$ and $Rad_g(\mathbb{Z}) \ll_g \mathbb{Z}$.

Proposition 2.19. Let μ be an indecomposable module. Then μ is a Rad_g-lifting module if and only if either μ is g-radical or μ is g-local.

Proof. ⟹) Let μ be an indecomposable Rad_g-lifting module. Assume $Rad_g(\mu) \neq \mu$. By Proposition 2.18, we have $Rad_g(\mu) \ll_g \mu$. Also, μ has an essential maximal submodule, say η, such that $Rad_g(\mu) \subseteq \eta$. By [8, Proposition 2.2(8)], there is a submodule γ of μ in η such that $\mu = \gamma \oplus \varrho$ and ϱ is a g-supplement of η in μ. Thus, $\mu = \eta + \varrho$ and $\eta \cap \varrho \ll_g \varrho$. As μ is an indecomposable module, so either $\varrho = 0$ or $\varrho = \mu$. If $\varrho = 0$ then, $\eta = \mu$, a contradiction. Hence $\varrho = \mu$ and so $\eta \ll_g \mu$. Then $\eta \subseteq Rad_g(\mu)$. Thus, $Rad_g(\mu) = \eta$. That means $Rad_g(\mu)$ is a maximal submodule of μ and $Rad_g(\mu) \ll_g \mu$. Therefore, μ is g-local.

$\Leftarrow$) If $Rad_g(\mu) = \mu$, [8, Proposition 2.3] implies that μ is a Rad_g-lifting module. Now, if μ is a g-local module, Proposition 2.2 implies μ is a Rad_g-lifting module. □

Corollary 2.20. Let μ be an indecomposable module with $Rad_g(\mu) \neq \mu$. If μ is a Rad_g-lifting module, then μ is a local.

Proof. By Proposition 2.19, μ is a g-local module. According to [6, Proposition 2.8], μ is local.

Corollary 2.21. Let μ be an indecomposable module with $Rad_g(\mu) \neq \mu$. If μ is a Rad_g-lifting module, then μ is hollow, and hence generalized hollow.

Proof. By Corollary 2.20, μ is local, and hence hollow, consequently, generalized hollow. □

Proposition 2.22. Let μ be an indecomposable module with $Rad_g(\mu) \neq \mu$. The assertions below are equivalent.

(1) μ is local.

(2) μ is hollow.

(3) μ is generalized hollow.

(4) μ is Rad_g-lifting.

(5) μ is g-local.

(6) μ is $sgrs^{\oplus}$-module.

Proof.

(1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) obvious.

(3) $\Longrightarrow$ (4) By [8, Proposition 2.13].

(4) $\Longrightarrow$ (5) By Proposition 2.19.

(5) $\Longrightarrow$ (1) By [6, Proposition 2.8].

(4) $\Longleftrightarrow$ (6) By Corollary 2.6. □

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Recall [15] that a module μ has summand sum property (or briefly, has SSP) if for every two direct summands ,then the sum of them is also direct summand.

Proposition 2.23. Let μ be a module that has SSP and $Rad_g(\mu) \leq^{\oplus} \mu$ and $Rad_g(\mu)$ is ⊕-g-supplemented module. If μ is a Rad_g-lifting, then μ is ⊕-g-supplemented.

Proof. Let $\eta \leq \mu$. Since $Rad_g(\mu) \subseteq Rad_g(\mu) + \eta$, so by [5, Proposition 2.2(8)], $Rad_g(\mu) + \eta$ has a g-supplement, say β, and $\beta \leq^{\oplus} \mu$. Now, as $Rad_g(\mu) \cap (\beta + \eta) \leq Rad_g(\mu)$ and since $Rad_g(\mu)$ is a ⊕-g-supplemented, then $Rad_g(\mu) \cap (\beta + \eta)$ has a g-supplement, say ϱ, and $\varrho \leq^{\oplus} Rad_g(\mu)$. Since $Rad_g(\mu) \leq^{\oplus} \mu$, it follows that $\varrho \leq^{\oplus} \mu$. As μ has SSP, we have $\beta + \varrho \leq^{\oplus} \mu$. By [4, Lemma 6], $\beta + \varrho$ is a g-supplement of η in μ. Thus μ is a ⊕-g-supplemented. □

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Conflict of interests.

There are non-conflicts of interest.

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