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Existence of Eigenvalues and Boundedness of the Eigenfunctions and the Orthogonality of the Eigenfunctions of a Type of Spectral Problem

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وجود القيم الذاتية وحدود الدوال الذاتية وتعامد الدوال الذاتية لنوع من المشاكل الطيفية

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ABSTRACT

Background:

This research aims to investigate certain requirements for the presence of eigenvalues, as well as the boundaries of eigenfunctions and their derivatives, specifically, the eigenfunctions' first as well as second derivatives.

Materials and Methods:

In this study, we use the spectral problem of second-order differential equations:

$$L[f] = -f''(x) + f'(x) + p(x)f(x) = \lambda r(x)f(x), x \in [0, \alpha],$$

with mixed boundary conditions

$$f'(\alpha) - i\,\lambda f(\alpha) = f'(0) - f(0) = 0, \text{ where } \lambda \text{ is a spectral parameter.}$$

and the normalized condition $\int_0^\alpha r(x)|f(x)|^2 dx = 1$, where $r(x) > 0$.

Results:

We get that the spectral parameter of a second-order differential operator is real. And we obtain Lagrange's identity for a spectral problem. Also, we prove that the spectral problem is self-adjoint, and the property of orthogonality of eigenfunctions is shown.

Conclusions:

In this research, we studied the existence of eigenvalues and the estimation of the norm of eigenfunctions for problem (1) - (3). Furthermore, we investigated the self-adjoint nature of the problem, and we proved that the eigenfunctions are orthogonal.

Keywords: self-adjoint; upper bound; spectral problem; spectral parameter; eigenfunctions

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INTRODUCTION

The area of spectral theory associated with determining the spectrum and eigenfunction expansion of a linear ordinary differential equation is known as the spectral theory of ordinary differential equations. A spectral problem of second-order differential equations was studied in [1-4]. It was possible to determine the upper bound of the eigenfunction, and the behavior of the eigenvalues and eigenfunctions of the boundary value problem with various boundary conditions has been considered in [1,3]. The Sturm-Liouville problem and its applications have also been investigated in [5-7]. Also, self-adjoint differential operators are studied in [8,9].In this work, I study some conditions for the existence of eigenvalues, the boundedness of eigenfunctions, and the upper bounds of the first and second derivatives of the eigenfunctions in relation to this spectrum problem:

$$L[f] = \lambda r(x) f(x), \quad x \in [0,\alpha], \tag{1}$$

$$f'(\alpha) - i\,\lambda f(\alpha) = f'(\alpha) - f(\alpha) = 0 \tag{2}$$

$$\int_0^\alpha r(x)|f(x)|^2 dx = 1 \tag{3}$$

Where $\lambda = \mu + iv$, $m \le r(x) \le M$, $0 < m \le M$.

Theorem 1. The inequality of problem (1) – (3) has been satisfied by the eigenfunction $\underset{x\in[0,\alpha]}{Max}|f(x)| = \|f(x)\| < K|\lambda|^{\frac{1}{4}}$, where λ is an eigenvalue, $\lambda = \mu + iv, \mu \ne 0$ and $r(x) \in L^+[0,a]$, $p(x) \in L[0,a]$, K is a positive number does not depend on r *and* p.

Proof. Considering every point x in $[0,\alpha]$, let's examine the identity

$$|f(x)|^2 = f(x)\bar{f}(x) = \int_0^x [\bar{f}(s) f'(s) + f(s)\bar{f}'(s)]ds + |f(0)|^2$$

$$= \int_0^x \frac{\sqrt{r(s)}.\,[\bar{f}(s)f'(s) + f(t)\bar{f}'(s)]}{\sqrt{r(s)}} ds + |f(0)|^2$$

From inequality $r(s) \ge m$, we obtain

$$|f(x)|^2 \le \int_0^x \frac{\sqrt{r(s)}\,|\bar{f}(s)f'(s) + f(s)\bar{f}'(s)|}{\sqrt{m}} ds + |f(0)|^2$$

$$\le \frac{1}{\sqrt{m}}\left[\int_0^x \sqrt{r(s)}\,|\bar{f}(s).f'(s)|\,ds + \int_0^x \sqrt{\rho(s)}\,|f(s).\bar{f}'(s)|\,ds\right] + |f(0)|^2$$

$$\le \frac{1}{\sqrt{m}}\left[\int_0^x \sqrt{r(s)}\ |\bar{f}(s)||f'(s)|\ ds + \int_0^x \sqrt{\rho(s)}\ |f(s)||\bar{f}'(s)|\,ds\right] + |f(0)|^2$$

$$= \frac{2}{\sqrt{m}} \int_0^x \sqrt{r(s)}\,|\bar{f}(s)||f'(s)|\,ds + |f(0)|^2$$

$$\leq \frac{2}{\sqrt{m}} \int_0^\alpha \sqrt{r(s)}\, |\bar{f}(s)||f'(s)|\, ds + |f(0)|^2$$

We determine that by applying the Cauchy-Schwartz inequality to the final integral.

$$|f(x)|^2 \leq \frac{2}{\sqrt{m}} \left[\int_0^\alpha r(s).|f(s)|^2 ds\right]^{1/2} \left[\int_0^\alpha |f'(s)|^2 ds\right]^{1/2} + |f(0)|^2$$

From the normalized condition (3) we get

$$|f(x)|^2 \leq \frac{2}{\sqrt{m}} \left[\int_0^\alpha |f'(s)|^2 ds\right]^{1/2} + |f(0)|^2 \qquad (4)$$

The equation (1) can be multiplied by $\bar{f}(x)$, and the resulting equation can be integrated from 0 up to α.

$$-\int_0^\alpha f''(x)\bar{f}(x)dx + \int_0^\alpha f'(x)\bar{f}(x)dx + \int_0^\alpha p(x)f(x)\bar{f}(x)dx = \int_0^\alpha \lambda r(x)f(x)\bar{f}(x)dx$$

With consideration for the boundary conditions (2), use the first integral by parts, we obtain

$$-i\lambda|f(\alpha)|^2 + |f(0)|^2 + \int_0^\alpha |f'(x)|^2 dx + \int_0^\alpha f'(x)\bar{f}(x)dx + \int_0^\alpha p(x)|f(x)|^2 dx =$$

$$\lambda \int_0^\alpha r(x)|f(x)|^2\, dx \qquad (5)$$

Equations (1)–(3) are now rewritten as follows:

$$-\bar{f}''(x) + \bar{f}'(x) + p(x)\bar{f}(x) = \bar{\lambda} r(x)\bar{f}(x), \quad x \in [0, \alpha], \qquad (6)$$

$$\bar{f}'(\alpha) - i\,\bar{\lambda}\bar{f}(\alpha) = \bar{f}'(0) - \bar{f}(0) = 0 \qquad (7)$$

$$\int_0^\alpha r(x)\left|\bar{f}(x)\right|^2 dx = 1 \qquad (8)$$

Solving equation (6) from 0 to α by multiplying it by $f(x)$ yields

$$-\int_0^\alpha \bar{f}''(x)f(x)dx + \int_0^\alpha \bar{f}'(x)f(x)dx + \int_0^\alpha p(x)\bar{f}(x)f(x)dx = \int_0^\alpha \bar{\lambda} r(x)\bar{f}(x)f(x)dx$$

Again, with consideration for the boundary conditions (2), use the first integral by parts, we obtain

$$i\bar{\lambda}|f(\alpha)|^2 + |f(0)|^2 + \int_0^\alpha |f'(x)|^2 dx + \int_0^\alpha \bar{f}'(x)f(x)dx + \int_0^\alpha p(x)|f(x)|^2 dx =$$

$$\bar{\lambda} \int_0^\alpha r(x)|f(x)|^2\, dx \qquad (9)$$

After multiplying equation (5) by $\bar{\lambda}$, and equation (6) by λ, we get the following by adding them

$$(\bar{\lambda}+\lambda)|f(0)|^2 + (\bar{\lambda}+\lambda)\int_0^\alpha |f'(x)|^2 dx + (\bar{\lambda}+\lambda)\int_0^\alpha \bar{f}'(x)f(x)dx + (\bar{\lambda}+\lambda)\int_0^\alpha p(x)|f(x)|^2 dx = (\bar{\lambda}+\lambda)\,|\lambda| \int_0^\alpha r(x)|f(x)|^2\, dx$$

Additionally, given that $\mu \neq 0$ then $\bar{\lambda} + \lambda \neq 0$ as a result

$$|f(0)|^2 + \int_0^\alpha |f'(x)|^2 dx + \int_0^\alpha \bar{f}'(x)f(x)dx + \int_0^\alpha p(x)|f(x)|^2 dx = |\lambda| \int_0^\alpha r(x)|f(x)|^2\, dx$$

$$\int_0^\alpha |f'(x)|^2 dx = |\lambda| \int_0^\alpha r(x)|f(x)|^2\, dx - |f(0)|^2 - \int_0^\alpha \bar{f}'(x)f(x)dx - \int_0^\alpha q(x)|f(x)|^2 dx$$

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And when we put this equation into equation (4), we get

$$|f(x)|^2 \leq \frac{2}{\sqrt{m}}\left[|\lambda| \int_0^\alpha r(x).|f(x)|^2 dx - |f(0)|^2 - \int_0^\alpha \bar{f}'(x) f(x) dx - \int_0^\alpha p(x)|f(x)|^2 dx\right]^{1/2} + |f(0)|^2,$$

$$|f(x)|^2 \leq \frac{2}{\sqrt{m}}\left[|\lambda| \int_0^\alpha r(x)|f(x)|^2 dx \left(1 - \frac{\left(|f(0)|^2 + \int_0^\alpha \bar{f}'(x)f(x)dx + \int_0^\alpha p(x)|f(x)|^2 dx\right)}{|\lambda| \int_0^\alpha r(x)|f(x)|^2 dx}\right)\right]^{1/2} + |f(0)|^2$$

Then

$$|f(x)|^2 \leq \frac{2\sqrt{|\lambda|}}{\sqrt{m}}\left[\int_0^\alpha r(x).|f(x)|^2 dx\right]^{1/2}\left[\left(1 - \frac{\left(|f(0)|^2 + \int_0^\alpha \bar{f}'(x)f(x)dx + \int_0^\alpha p(x)|f(x)|^2 dx\right)}{|\lambda| \int_0^\alpha r(x)|f(x)|^2 dx}\right)\right]^{1/2} + |f(0)|^2$$

From equation (3), and for large $|\lambda|$ we get

$$|f(x)|^2 \leq \frac{2\sqrt{|\lambda|}}{\sqrt{m}} + |f(0)|^2$$

$$|f(x)|^2 \leq \sqrt{|\lambda|}\left(\frac{2}{\sqrt{m}} + \frac{|f(0)|^2}{\sqrt{|\lambda|}}\right)$$

$$|f(x)| \leq |\lambda|^{\frac{1}{4}}\left(\frac{2}{\sqrt{m}} + \frac{|f(0)|^2}{\sqrt{|\lambda|}}\right)^{\frac{1}{2}}$$

$$\underset{x\in[0,\alpha]}{Max}|f(x)| = \|f(x)\| < K|\lambda|^{\frac{1}{4}}, \text{ where } K = \left(\frac{2}{\sqrt{m}} + \frac{|f(0)|^2}{\sqrt{|\lambda|}}\right)^{\frac{1}{2}}.$$

Lemma1. For the existence of eigenvalues of the problem (1) - (3) must:

1. The formula $mv(i - (f(0))^2) > 0$ is true if $\mu = 0$.
2. The formula $\mu - iv - k^2 - t - \frac{m}{4}(i - k^2)^2 \geq \frac{m}{M}$ is true if $\mu \neq 0$.

Proof. If $\mu = 0$, then (1) - (3) reduces to

$$-f''(x) + f'(x) + p(x)f(x) = ivr(x)f(x) \quad (10)$$

$$f'(\alpha) + vf(\alpha) = 0 = f'(0) - f(0) \quad (11)$$

$$\int_0^\alpha r(x)|f(x)|^2 dx = 1 \quad (12)$$

Equation (10) is multiplied by $f(x)$, and the end result is integrated from 0 to α.

$$-\int_0^\alpha f''(x)\, f(x)dx + \int_0^\alpha f'(x)\, f(x)dx + \int_0^\alpha p(x)\, f^2(x)dx = iv\int_0^\alpha r(x)\, f^2(x)dx$$

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Integrate the first integral by parts

$$v\big(f(\alpha)\big)^2 + \big(f(0)\big)^2 + \int_0^\alpha [f'(x)]^2\,dx + \int_0^\alpha f'(x)\,f(x)dx + \int_0^\alpha p(x)\,f^2(x)dx = iv$$

$$v\big(f(\alpha)\big)^2 = iv - \big(f(0)\big)^2 - \int_0^\alpha [f'(x)]^2\,dx - \int_0^\alpha f'(x)\,f(x)dx - \int_0^\alpha p(x)\,f^2(x)dx \qquad (13)$$

Additionally, since we get from Theorem 1

$$|f(x)|^2 \le \frac{2}{\sqrt{m}}\left[\int_0^\alpha |f'(x)|^2 dx\right]^{1/2} + |f(0)|^2$$

$$\frac{\sqrt{m}}{2}(|f(\alpha)|^2 - |f(0)|^2) \le \left[\int_0^\alpha |f'(x)|^2 dx\right]^{1/2}$$

$$\left[\frac{\sqrt{m}}{2}(|f(\alpha)|^2 - |f(0)|^2)\right]^2 \le \int_0^\alpha |f'(x)|^2 dx$$

From the inequality and equation (13)

$$v\big(f(\alpha)\big)^2 \ge iv - \big(f(0)\big)^2 - \left[\frac{\sqrt{m}}{2}(|f(\alpha)|^2 - |f(0)|^2)\right]^2 - \int_0^\alpha f'(x)\,f(x)dx - \int_0^\alpha p(x)\,f^2(x)dx$$

$$v\big(f(\alpha)\big)^2 \ge iv - \big(f(0)\big)^2 - \left[\frac{\sqrt{m}}{2}(|f(\alpha)|^2 - |f(0)|^2)\right]^2 - \int_0^\alpha f'(x)\,f(x)dx - \int_0^\alpha \frac{p(x)}{r(x)} r(x) f^2(x)dx$$

$$v\big(f(\alpha)\big)^2 \ge iv - \big(f(0)\big)^2 - \frac{m}{4}|f(\alpha)|^4 + \frac{m}{2}|f(\alpha)|^2|f(0)|^2 - \frac{m}{4}|f(0)|^4 - \int_0^\alpha f'(x)\,f(x)dx - \frac{m}{M} \qquad (14)$$

If we denote $\big(f(\alpha)\big)^2$ by z, then (14) reduces to

$$vz \ge iv - \big(f(0)\big)^2 - \frac{m}{4}z^2 + \frac{m}{2}z|f(0)|^2 - \frac{m}{4}|f(0)|^4 - \int_0^\alpha f'(x)\,f(x)dx - \frac{m}{M}$$

$$-\frac{m}{4}z^2 + \left(\frac{m}{2}|f(0)|^2 - v\right)z + (iv - \big(f(0)\big)^2 - \frac{m}{4}|f(0)|^4 - \int_0^\alpha f'(x)\,f(x)dx - \frac{m}{M}) \le 0$$

This can only occur if the discriminating

$$D = \left(\frac{m}{2}|f(0)|^2 - v\right)^2 + m(iv - \big(f(0)\big)^2 - \frac{m}{4}|f(0)|^4 - \int_0^\alpha f'(x)\,f(x)dx - \frac{m}{M}) > 0$$

Or $v^2 + \left(im - m\big(f(0)\big)^2\right)v - (m|f(0)|^2 + m\int_0^\alpha f'(x)\,f(x)dx + \frac{m^2}{M}) > 0$

And therefore $mv(i - \big(y(0)\big)^2) > 0$

Consider the case $\mu \neq 0$.

To obtain the relation, multiply equation (1) by $\bar{f}(x)$, and the adjoint equation (6) by $f(x)$. Then, add the two resulting equations.

$$-\left[f''(x)\bar{f}(x)+\bar{f}''(x)f(x)\right]+\left[f'(x)\bar{f}(x)+\bar{f}'(x)f(x)\right]+2p(x)|f(x)|^2 = (\lambda+\bar{\lambda})r(x)|f(x)|^2$$

By integrating the two sides of this equation, we arrive at

$$-\int_0^\alpha\left[f''(x)\bar{f}(x)+\bar{f}''(x)f(x)\right]dx+\int_0^\alpha\left[f'(x)\bar{f}(x)+\bar{f}'(x)f(x)\right]dx+2\int_0^\alpha p(x)|f(x)|^2dx = (\lambda+\bar{\lambda})\int_0^\alpha r(x)|f(x)|^2\,dx$$

With consideration for the boundary conditions (2), use the first integral by parts we obtain

$$i(\bar{\lambda}-\lambda)|f(\alpha)|^2+2|f(0)|^2+2\int_0^\alpha|f'(x)|^2dx+\int_0^\alpha\left[f'(x)\bar{f}(x)+\bar{f}'(x)f(x)\right]dx+2\int_0^\alpha p(x)|f(x)|^2dx=(\lambda+\bar{\lambda})\int_0^\alpha r(x)|f(x)|^2\,dx$$

Hence using the normalizing condition (3) and the equality $\lambda+\bar{\lambda}=2\mu$, $i(\bar{\lambda}-\lambda)=2\nu$, and by subtracting equation (5) from equation (9) we find $|f(\alpha)|^2=i$

$$2i\nu+2|f(0)|^2+2\int_0^\alpha|f'(x)|^2dx+2\int_0^\alpha\left[f'(x)\bar{f}(x)+\bar{f}'(x)f(x)\right]dx+2\int_0^\alpha p(x)|f(x)|^2dx=2\mu$$

or

$$i\nu+|f(0)|^2+\int_0^\alpha|f'(x)|^2dx+\int_0^\alpha\left[f'(x)\bar{f}(x)+\bar{f}'(x)f(x)\right]dx+\int_0^\alpha p(x)|f(x)|^2dx=\mu$$

$$\int_0^\alpha|f'(x)|^2dx+\int_0^\alpha p(x)|f(x)|^2dx=\mu-i\nu-|f(0)|^2-\int_0^\alpha\left[f'(x)\bar{f}(x)+\bar{f}'(x)f(x)\right]dx,$$

Since,

$$\int_0^\alpha p(x)|y(x)|^2dx=\int_0^\alpha\frac{p(x)}{r(x)}r(x)\,|f(x)|^2dx\geq\frac{m}{M}\int_0^\alpha r(x)\,|f(x)|^2dx=\frac{m}{M},$$

Then $\mu-i\nu-|f(0)|^2-\int_0^\alpha\left[f'(x)\bar{f}(x)+\bar{f}'(x)f(x)\right]dx\geq\int_0^\alpha|f'(x)|^2dx+\frac{m}{M}$ (15)

In Theorem 1 Shown, that

$$|f(\alpha)|^2\leq\frac{2}{\sqrt{m}}\left(\int_0^\alpha r(x)\,|f(x)|^2dx\right)^{\frac{1}{2}}\left(\int_0^\alpha|f'(x)|^2\,dx\right)^{\frac{1}{2}}+|f(0)|^2$$

and therefore

$$i\leq\frac{2}{\sqrt{m}}\left(\int_0^\alpha|f'(x)|^2\,dx\right)^{\frac{1}{2}}+|f(0)|^2$$

$$i-|f(0)|^2\leq\frac{2}{\sqrt{m}}\left(\int_0^\alpha|f'(x)|^2\,dx\right)^{\frac{1}{2}}$$

$$(i-|f(0)|^2)^2\leq\frac{4}{m}\int_0^\alpha|f'(x)|^2\,dx$$

$\frac{m}{4}(i-|f(0)|^2)^2\leq\int_0^\alpha|f'(x)|^2\,dx$, by substitution in equation (15) we get

$$\mu-i\nu-|f(0)|^2-\int_0^\alpha\left[f'(x)\bar{f}(x)+\bar{f}'(x)f(x)\right]dx\geq\frac{m}{4}(i-|f(0)|^2)^2+\frac{m}{M}$$

let $|f(0)|=k$, $\int_0^\alpha\left[f'(x)\bar{f}(x)+\bar{f}'(x)f(x)\right]dx=t$

$$\mu-i\nu-k^2-t-\frac{m}{4}(i-k^2)^2\geq\frac{m}{M}.$$

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Theorem 2. Let us consider that the functions $p(x)$ and $r(x)$ have an integrable on the closed interval $[0,\alpha]$, where $p(x) \geq 0$ and $r(x) \geq m > 0$ Then, all of the eigenvalues and related eigenfunctions of the problem (1) - (3) have positive constants c_1 and c_2, and they are independent on $p(x)$ and $r(x)$, therefore the inequality that follows is true

$$\|f'(x)\|_{C_{[0,\alpha]}} < c_1|\lambda|^{\frac{1}{4}};$$

$$\|f''(x)\|_{C_{[0,\alpha]}} < c_2|\lambda|^{\frac{5}{4}}.$$

Proof. Take into consideration the following identity:

$$|f'(x)|^2 = f'(x)\cdot \bar{f}'(x) = \int_0^x [f''(s)\bar{f}'(s) + f'(s)\bar{f}''(s)]ds + |f'(0)|^2,$$

$$\leq \int_0^x |f''(s)\bar{f}'(s) + f'(s)\bar{f}''(s)|\, ds + |f'(0)|^2$$

$$\leq 2\int_0^\alpha |f'(s)f''(s)|\, ds + |f(0)|^2$$

$$\leq 2\int_0^\alpha |f'(s)||f''(s)|\, ds + |f(0)|^2$$

(since $|f''(s)| = |\bar{f}''(s)|, |f'(s)| = |\bar{f}'(s)|,\ f'(0) - f(0) = 0$).

Estimating the integral by Cauchy- Schwartz inequality

$$|f'(x)|^2 \leq 2\left(\int_0^\alpha |f'(s)|^2 ds\right)^{\frac{1}{2}}\left(\int_0^\alpha |f''(s)|^2 ds\right)^{\frac{1}{2}} + |f(0)|^2$$

$$|f'(x)|^2 \leq 2\left(|\lambda|\int_0^\alpha r(x).|f(x)|^2 dx - |f(0)|^2 - \int_0^\alpha \bar{f}'(x)f(x)dx - \int_0^\alpha p(x)|f(x)|^2 dx\right)^{\frac{1}{2}} A + |f(0)|^2.$$

Where $A = \left(\int_0^\alpha |f''(s)|^2 ds\right)^{\frac{1}{2}}$ is positive real number

$$|f'(x)|^2 \leq 2\left[|\lambda|\int_0^\alpha r(x)|f(x)|^2\, dx\left(1 - \frac{\left(|f(0)|^2 + \int_0^\alpha \bar{f}'(x)f(x)dx + \int_0^\alpha p(x)|f(x)|^2 dx\right)}{|\lambda|\int_0^\alpha r(x)|f(x)|^2 dx}\right)\right]^{1/2} A + |f(0)|^2$$

$$|f'(x)|^2 \leq 2|\lambda|^{\frac{1}{2}} A + |f(0)|^2$$

$$|f'(x)|^2 \leq |\lambda|^{\frac{1}{2}}\left(2A + \frac{|f(0)|^2}{|\lambda|^{\frac{1}{2}}}\right)$$

$$|f'(x)| \leq c_1|\lambda|^{\frac{1}{4}}\text{ ,where } c_1 = \sqrt{2A + \frac{|f(0)|^2}{|\lambda|^{\frac{1}{2}}}}$$

$$Max|f'(x)| \leq c_1|\lambda|^{\frac{1}{4}}$$

$$\|f'(x)\| \leq c_1|\lambda|^{\frac{1}{4}}.$$

The latter part needs to be proven. We obtain from equation (1)

$$|f''(x)| = |\lambda r(x)f(x) - f'(x) - p(x)f(x)|$$

$$\leq |\lambda r(x)f(x)| + |f'(x)| + |p(x)f(x)|$$

$$\leq |\lambda||r(x)|\max|f(x)| + \max|f'(x)| + |p(x)|\max|f(x)|$$

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$$\leq |\lambda| Mk|\lambda|^{\frac{1}{4}} + c_1|\lambda|^{\frac{1}{4}} + Mk|\lambda|^{\frac{1}{4}}$$
$$= |\lambda|^{\frac{5}{4}} Mk + c_1|\lambda|^{\frac{1}{4}} + Mk|\lambda|^{\frac{1}{4}}$$
$$= |\lambda|^{\frac{5}{4}}(Mk + c_1|\lambda| + Mk|\lambda|)$$
$$= c_2|\lambda|^{\frac{5}{4}}, \text{ where } c_2 = (Mk + c_1|\lambda| + Mk|\lambda|).$$

Theorem 3. The problem (1), (2) is self-adjoint problem.
Proof. Let u and v be functions with second derivatives that are continuous on the interval $0 \leq x \leq \alpha$. Then $\int_0^\alpha L[u]\, v dx = \int_0^\alpha [-u'' + u' + p(x)u]\, v dx$.
Integrating the first expression on the right side two times by parts, we obtain

$$\int_0^\alpha L[u]\, v dx = -u'(x)v(x)|_0^\alpha + u(x)v'(x)|_0^\alpha - \int_0^\alpha uv''\, dx + \int_0^\alpha u'\, v dx + \int_0^\alpha pu\, v dx$$
$$=-[u'(x)v(x) - u(x)v'(x)]|_0^\alpha + \int_0^\alpha uL[v]\, dx$$

Hence, $\int_0^\alpha \{L[u]v - uL[v]\}\, dx = -[u'(x)v(x) - u(x)v'(x)]|_0^\alpha$, (16)
This is Lagrange's identity.
Now, let us assume that the function u and v in equation (16) also satisfies the boundary conditions. Following that, equation (16)'s right side becomes
$$\int_0^\alpha \{L[u]v - uL[v]\}\, dx = 0.$$
So Lagrange's identity in equation (16) reduces to
$$\int_0^\alpha \{L[u]v - uL[v]\}\, dx = 0. \quad (17)$$
Two real valued functions u and v in the interval $0 \leq x \leq \alpha$ have an inner product that is
$$(u, v) = \int_0^\alpha u(x)\, v(x) dx \quad (18)$$
So equation (17) becomes
$$(L[u], v) - (u, L[v]) = 0 \quad (19)$$
Theorem 4. There are no complex eigenvalues for the self-adjoint problem (1), (2).
Proof. Assume that φ is the corresponding complex eigenfunction, and λ is the complex eigenvalue of the problem (1),(2).
In equation (19) we have
$$(L[\varphi], \varphi) = (\varphi, L[\varphi])$$
However, we know that $L[\varphi] = \lambda r\varphi$
So $$(\lambda r\varphi, \varphi) = (\varphi, \lambda r\varphi).$$
By using the definition of inner product of complex-valued function
$$\int_0^\alpha \lambda r(x)\varphi(x)\bar{\varphi}(x)\, dx = \int_0^\alpha \varphi(x)\bar{\lambda}\bar{r}(x)\bar{\varphi}(x)\, dx$$
Since $r(x)$ is real, so we have
$$(\lambda - \bar{\lambda}) \int_0^\alpha r(x)\varphi(x)\bar{\varphi}(x) dx = 0,$$

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or

$$(\lambda-\bar{\lambda})\int_0^\alpha r(x)|\varphi(x)|^2dx=0,$$

Since the integrand is non-negative and not identically zero, therefore $\lambda-\bar{\lambda}$ must be zero.
We conclude that the problem can have no complex eigenvalues.

Theorem 5. Let λ_1 and λ_2 be distinct eigenvalues of the boundary value problems (1), (2), and let $\varphi_1(x)$ and $\varphi_2(x)$ be related eigenfunctions. Then $\varphi_1(x)$ and $\varphi_2(x)$ are orthogonal about the interval $(0,\alpha)$ and the weight function $r(x)$.

Proof. We note that the functions $\varphi_1(x)$ and $\varphi_2(x)$ satisfy the differential equation (1)

$$L[\varphi_1]=\lambda_1 r\varphi_1$$

and

$$L[\varphi_2]=\lambda_2 r\varphi_2$$

So substitute $L[\varphi_1]$ and $L[\varphi_2]$ in equation (19) we obtain

$$(\lambda_1 r\varphi_1,\varphi_2)-(\varphi_1,\lambda_2 r\varphi_2)=0,$$

or, using the definition of inner product of complex-valued function

$$\lambda_1\int_0^\alpha r(x)\varphi_1(x)\overline{\varphi_2}(x)\,dx-\overline{\lambda_2}\int_0^\alpha \varphi_1(x)\bar{r}(x)\overline{\varphi_2}(x)\,dx=0.$$

Because $\lambda_2, r(x)$, and $\varphi_2(x)$ are real, so this equation becomes

$$(\lambda_1-\lambda_2)\int_0^\alpha r(x)\varphi_1(x)\varphi_2(x)dx=0.$$

Since $\lambda_1\neq\lambda_2$, so $\int_0^\alpha r(x)\varphi_1(x)\varphi_2(x)dx=0.$

CONCLUSION

I provide an overview of the spectral theory of a second-order differential operator on a finite interval. I illustrated the norm of eigenfunctions, and I also proved the upper bounds for the first and second derivatives of eigenfunctions. Furthermore, the boundary value problem is self-adjoint; it has no complex eigenvalues. Also, I showed that the eigenfunctions corresponding to the eigenvalues are orthogonal.

Conflict of interests.

There are non-conflicts of interest.

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الخلاصة

مقدمة:

يهدف هذا البحث إلى التحقق من متطلبات معينة لوجود القيم الذاتية، وكذلك حدود الدوال الذاتية ومشتقاتها، وتحديداً مشتقات الدوال الذاتية الأولى والثانية.

طرق العمل:

استخدمنا في هذه الدراسة المسألة الطيفية للمعادلات التفاضلية من الدرجة الثانية:

$L[f] = -f''(x) + f'(x) + p(x)f(x) = \lambda r(x)f(x), x \in [0, \alpha]$،

مع شروط الحدود المختلطة

$f'(\alpha) - i\,\lambda f(\alpha) = f'(0) - f(0) = 0$، حيث λ هي البارامترات الطيفية

والحالة الطبيعية $\int_0^\alpha r(x)|f(x)|^2 dx = 1$,حيث $r(x) > 0$

الاستنتاجات:

لقد حصلنا على أن المعلمة الطيفية للمشغل التفاضلي من الدرجة الثانية حقيقية. وحصلنا على هوية لاغرانج للمسألة الطيفية. كما أثبتنا أن المشكلة الطيفية متجاورة ذاتياً، وتم إظهار خاصية تعامد الدوال الذاتية.

الكلمات المفتاحية: ذاتية مجاورة، الحد الأعلى، المشكلة الطيفية، المعلمة الطيفية، الوظائف الذاتية