Generalization of Tichonov and Hausdorff Separation Axiomes in Intuitionistic Fuzzy Special Topological Spaces

Fatimah M. Mohammed **Younis J. Yaseen**
Department of Mathematics, College of Education for Pure Sciences,
Tikrit University, Tikrit, IRAQ.
nafea_y2011@yahoo.com

Abstract
Our goal in this paper is to give definitions of generalized of Tichonov and Hausdorff separation axioms in intuitionistic fuzzy special topological spaces, and study relationships between these spaces with the intuitionistic special topological spaces $(X, \tau_{0,1})$ and $(X, \tau_{0,2})$ on one hand and intuitionistic fuzzy special topological spaces (X, τ_1) and (X, τ_2) on the other hand.

Keywords: intuitionistic fuzzy special topological spaces**,** $(X, \tau_{0,1})$space, $(X, \tau_{0,2})$ space**,** (X, τ_1) space, (X, τ_2) space.

الخلاصة
هدفنا من هذا البحث هو تعميم تعريف بديهية فصل تيكنوف وبديهية الفصل هاوسدورف في الفضاءات التبولوجية الحدسية الخاصة ودراسة العلاقات بين هذه الفضاءات مع الفضاءات التبولوجية الحدسية الخاصة من نوع (X, τ_2) و (X, τ_1) من جهة والفضاءات التبولوجية الحدسية الخاصة من نوع $(X, \tau_{0,2})$و $(X, \tau_{0,1})$ من جهة اخرى.

الكلمات المفتاحية: الفضاءات التبولوجية الحدسية الخاصة، فضاء$(X, \tau_{0,1})$ فضاء $(X, \tau_{0,2})$فضاء (X, τ_1) , فضاء (X, τ_2).

1.Introduction
The term of fuzzy sets studied by many authors. Firstly, [Zadeh, 1965] had introduced it, then [Chang, 1968], introduced the concept of fuzzy topological space. Later, as an extension of Zadeh's study of fuzzy sets, [Atanassov, 1986] the concept of intuitionistic fuzzy sets was introduced. Later, [Coker, 1997] defined the topology of intuitionistic fuzzy sets.

In this paper we study the new concepts of Tichonov and Hausdorff separation axioms in intuitionistic fuzzy special topological spaces and generalized it with the spaces $(X, \tau_{0,1})$, $(X, \tau_{0,2})$, (X, τ_1) and (X, τ_2) on the intuitionistic fuzzy special topological spaces.

2. Preliminaries
Firstly, we shall present the fundamental definitions. Let X be a non-empty fixed set. An intuitionistic fuzzy special sets (IFSS, for short) and A is an object having the form $< x, A_1, A_2 >$, where A_1 and A_2 are sub set of X and satisfying the following condition $A_1 \cap A_2 = \emptyset$. Whenever, the set A_1 is called the set of member of A, while A_2 is called the set of non-member of A. Every subset A of a non-empty set

X is an IFSS having the form < x, A, A^c > where A^c refers to the complement of the set A.

[Coker, 1987] defined some Boolean algebra operations in an IFSS as followes:

Let X be any non-empty set and an IFS' A, B where < x, A_1, A_2 >, B = < x, B_1, B_2 > respectively. Furthermore, let {A_i: i ∈ J} be an arbitrary family of IFSS in X. Then:

(i). $A \subseteq B \leftrightarrow A_1 \subseteq B_1$ and $A_2 \supseteq B_2$;

(ίί) $A = B \leftrightarrow A \subseteq B$ and $A \supseteq B$;

(ίίί) $A^c = < x, A_2, A_1 >$;

(iv) FA= < x,A_1,$A^c{}_1$ > , SA = < x, $A^c{}_2$, A_2 > ;

(v) $\cup A_ί = < x, \cup A_ί^{(1)}, \cap A_ί^{(2)} >$, $\cap A_ί = < x, \cap A_ί^{(1)}, \cup A_ί^{(2)} >$;

(vί) $\tilde{\emptyset} = < x, \emptyset, X>$, $\tilde{X} = < x, X, \emptyset >$.

Recall that an intuitionistic fuzzy special topology (IFTS, for short) on a non-empty set X is a family τ of IFSS in X containing $\tilde{\emptyset}$, $\tilde{X}$ and closed under finite intersection and arbitrary union. So, in this case the pair (X, τ) is called an intuitionistic fuzzy special topological space (IFSTS, for short).

Let (X, τ) be an IFSTS, we can construction two IFSTS' s as follow as:

$\tau_{0,1}$ = { FG: G ∈ τ } and $\tau_{0,2}$ = { SG: G ∈ τ }. Also, the two topological spaces

τ_1 = {G_1: G = < x,G_1, G_2 > ∈ τ } and τ_2 = {$G_2{}^c$: G = < x,G_1, G_2 > ∈ τ }.

The element x in an IFSS is belong to A is define by:

$x \in A = < x, A_1, A_2 > \Leftrightarrow x \in A_1$ and $x \notin A_2$.

3. Tichonov Separation Axioms on Intuitionistic Fuzzy Special Topological Spaces.

In this section, we introduced Tichonov separation axiom on IFSTS.

Definition 3.1:

Let (X, τ) be an IFSTS, then (X, τ) is said to be satisfy Tichonove separation axioms if (Tich., for short) if for each x, y ∈ X and x ≠ y, there exist A and B ∈ τ such

That x ∈ A, y ∈ B and x ∉ B, y ∉ A.

Theorem 3.2

Let (X, τ) be an IFSTS, then the following are equivalent:

(i) (X, τ) is a Tich. Space;

(ii) (X, $\tau_{0,1}$) is a Tich. Space;

(iii) (X, τ_1) is a Tich. Space.

Proof : (i) ⟹ (ii)

Let x, y $\in$ X such that x $\neq$ y, then there exist U_x = < x, A_1, A_2 >, V_y = < y, B_1, B_2 > such that x $\in$ U_x , y $\in$ V_y , x $\notin$ V_y and y $\notin$ U_x.
Thus, x $\in$ A_1, x $\notin$ A_2 and y $\in$ B_1, y $\notin$ B_2, also, x $\notin$ B_1, y $\notin$ A_1.
Since, FU_X = < x,A_1,A^c_1 > and FV_y = < x, B_1, B^c_1 >, then x $\in$ A_1 and x $\notin$ A^c_1. So, x $\in$ FU_X , x $\notin$ V_y $\Longrightarrow$ x $\notin$ B_1 or x $\in$ B_2. Now, if x $\notin$ B_1 then x $\in$ B_1^c.

Therefore, x $\notin$ FV_y. If x $\in$ B_2, then x $\notin$ B_2 and since $B_1 \cap B_2 = \emptyset$. So, x $\in$ B_1^c. Thus, x $\notin$ FV_y. Similarly, y $\in$ FV_y and x $\notin$ FV_y. Therefore, (X, $\tau_{0,1}$) is a Tich. Space.

(ii) $\Longrightarrow$ (iii) Suppose that x, y $\in$ X such that x $\neq$ y, then there exist FU_x = < x, A_1, A_1^c > and FV_y = < y, B_1, B_1^c > in $\tau_{0,1}$ where, exist U_x = < x, A_1, A_2 > and V_y = < y, B_1, B_2 > in τ such that x $\in$ FU_x , y $\in$ FV_y , x $\notin$ FV_y and y $\notin$ FU_x.
Thus, x $\in$ A_1 and not in B_1 and y in B_1 not in A_1. Therefore, (X, τ_1) is a Tich. Space.

(iii) $\Longrightarrow$ (i) Let x, y $\in$ X such that x $\neq$ y, then there exist x $\in$ A_1 and x $\notin$ B_1 and y $\in$ B_1, y $\notin$$A_1$ where A_1 and B_1 are in τ_1. Put, U_x = < x, A_1, A_2 >, V_y = < y, B_1, B_2 >. So, U_x and V_y are in τ and satisfy Tich. Axiom, therefore (X, τ) is a Tich. Space.

Proposition 3.3

Let (X, τ) be a Tich. IFSTS, then,

(i). (X, $\tau_{0,2}$) is a Tich. space;

(ii). (X, τ_2) is a Tich. space.

Proof: (i)

Let (X, τ) be a Tich. IFSTS, and let x and y be any two elements in X such that x $\neq$ y**,** then there exists exist U_x = < x, A_1, A_2 >, V_y = < y, B_1, B_2 > such that x $\in$ U_x , y $\in$ V_y , x $\notin$ V_y and y $\notin$ U_x. Thus, x $\in$ A_1, x $\notin$ A_2 and y $\in$ B_1, y $\notin$ B_2, also, x $\notin$ B_1, y $\notin$ A_1.

Since**,** SU_x = < x, A_2^c, A_2 > and SV_y = < x, B_2^c, B_2 >**,** so x $\in$ A_2^c and **y** $\in$ B_2^c**,** thus x $\in$ SU_x and y $\in$ SV_y .

Similarly, we can show that, x $\notin$ SV_y and y $\notin$ SU_x**.** Therefore, (X, $\tau_{0,2}$) is a Tich. space; in similar way we can prove, (ii).

Remark 3.4

The converse of proposition 3.3 is not true in general. The following example show the case.

Example 3.5

Let X = {a, b, c} and A = < x, $\emptyset$, {c} >, B= < x, {b}, {a}> and C = < x, {c}, {b} >.
Put $\tau = \{\widetilde{\emptyset}, \tilde{X}$, A, B, C, A $\cup$ C, A $\cup$ B, B $\cup$C, A $\cap$ B, A $\cap$ C, B $\cap$ C$\}$. So, we can see easily that (X, $\tau_{0,2}$) and (X, τ_2) is Tich. spaces, but not (X, τ).

Remark 3.6

(i). If (X, τ_2) is Tich. space**,** then (X, τ_1) is not necessary to be Tich. space.

(ii). If (X, τ_1) is Tich. space, then (X, τ_1) is not.
Example 3.5 shows (i) and the example 3.7 below shows(ii).

Example 3.7

Let X = {a, b, c} and A = < x, {a}, {b} >, B= < x, {b}, {a}> and C = < x, {c}, {a, b} >. Define, $\tau = \{\widetilde{\emptyset}, \tilde{X}$, A, B, C, A ∪ C, A ∪ B, B ∪C, A ∩ B, A ∩ C, B ∩ C}. So, we can see very easily that (X, τ_1) a is Tich. space, but (X, τ_2) is not.

4. Hausdorff Space on Intuitionistic Fuzzy Special Topological spaces.

In this section, we introduced Hausdorff separation axiom on IFSTS.

Definition 4.1

Let (X, τ) be an IFSTS, then (X, τ) is said to be satisfy Hausdorff separation axioms (Haus., for short) if for each x and y in the set X such that $x \neq y$, there exist A, B ∈ τ such that x ∈ A and y ∈ B, $A \cap B = \widetilde{\emptyset}$.

Proposition 4.2

Let (X, τ) be a Haus. IFSTS, then:
(i). (X, $\tau_{0,1}$) is a Haus. space;
(ii). (X, τ_1) is a Haus. space;
(iii). (X, $\tau_{0,2}$) is a Haus. space;
(ii). (X, τ_2) is a Haus. space.

Proof: direct.

Remark 4.3

The converse of proposition 4.2 is not true in general. The following example show the cases.

Example 4.

Let X = {a, b, c} and A = < x, {a}, {c} >, B= < x, {c}, {b}> and C = < x, {b}, {a } >. Define, $\tau = \{\widetilde{\emptyset}, \tilde{X}$, A, B, C, A ∪ C, A ∪ B, B ∪C, A ∩ B, A ∩ C, B ∩ C}. So, we can see very easily that τ is not Haus.. space, but (X, $\tau_{0,1}$) and (X, τ_2) are Haus. space.

Example 4.5

Let X = {a, b, c, d} and A = < x, {d}, {b, c} >, B = < x, {c}, {a, b}>, C = < x, {b}, {a, d } > and D = < x, {a}, {c, d} >. Define, $\tau = \{\widetilde{\emptyset}, \tilde{X}$, A, B, C} ∪ {all pair wise
intersection and unioin}. So, we can see that (X, τ) is not Haus.. space, but (X, $\tau_{0,2}$) is Haus. space.

Example 4.6

Let X = {a, b, c } and A = < x, {a}, {b } >, B = < x, {b}, {a }> and C = < x, {c}, {a, c } > . Define, $\tau = \{\tilde{\emptyset}, \tilde{X}$, A, B, C A ∪ C, B ∪C, A ∩ B, A ∩ C}. So, we can see that τ is not Haus. space, but (X, τ_1) is Haus. space.

Proposition 4.7

Let (X, τ) be an IFSTS such that (X, $\tau_{0,1}$), ((X, $\tau_{0,2}$)) is Haus. space, then (X, τ) is also Haus. space.

Proof:

Let (X, $\tau_{0,1}$) be a Haus. space, then for each x, y in X such that x ≠ y**,** then there exist A = < x, A_1, A_2 >, B = < y, B_1, B_2 > ∈ τ and x ∈ FA , y ∈ FB, FA ∩ FB = $\tilde{\emptyset}$.
So, x in A and y in B such that A and B are disjoint i.e (X, τ) is Haus. space.
Similarly, we can prove (X, τ) is Haus. space when (X, $\tau_{0,2}$) is Haus. space.

Proposition 4.8

Let (X, τ_1) be a Haus. space, then for each x, y in X such that (X, τ_1), ((X, τ_2)) is Haus. space, then (X, τ) is also Haus. space.

Proof:

Let (X, $\tau_{0,1}$) be a Haus. space, then for each x, y in X such that x ≠ y**,** then there exist A = < x, A_1, A_2 >, B = < y, B_1, B_2 > ∈ τ such that x ∈ A_1 , y ∈ B_1 and $A_1 \cap B_1$
$= \tilde{\emptyset}$. So, x in A and y in B whenever, A and B are disjoints i.e (X, τ) is a Haus. space.

Now, to prove (X, τ) is a Haus. space when (X, τ_2) is a Haus. space, let x, y in X such that x ≠ y**,** so there exist A = < x, A_1, A_2 >, B = < y, B_1, B_2 > ∈ τ such that x ∈
A_2^c , y ∈ B_2^c and $A_2^c \cap B_2^c = \tilde{\emptyset}$. Then $A_2 \cup B_2 = \tilde{X}$ and we have $A_1 \cap A_2 = \tilde{\emptyset}$, $B_1 \cap B_2 =$
$\tilde{\emptyset}$ then, from this we can use the set theory operation to get,

$$[(A_1 \cap A_2) \cap B_1] \cup [(A_1 \cap A_2) \cap B_2] = \tilde{\emptyset}.$$

And $A_1 \cap B_1 = \tilde{\emptyset}$. So, $A \cap B = \tilde{\emptyset}$. i.e. (X, τ) is also Haus. space.

Remark 4.9

Hasdroff property is independent between (X, τ_1) and (X, τ_2) see Example 3.5 and Example 4.6.

Remark 4.10

Every Huas. space is Tich. space, but the converse is not true in general.

Example 4.11

Let $X = \{a, b\}$ and $A = < x, \{a\}, \{b\} >$ and $B = < x, \{b\}, \emptyset >$. Define, $\tau = \{\tilde{\emptyset}, \tilde{X}, A, B, A \cap B\}$. Then, we can see that (X, τ) is a Tich. space, but not Haus. space.

References

Atanassov K., 1986, " Intuitionistic fuzzy sets", *Fuzzy sets and System*, 20 (1), 87-96.

Chang C. L., 1968, "Fuzzy topological spaces", *J. Math. Anal. Appl.*, 24, 182-190.

Coker, D., 1997, "An introduction to intuitionistic fuzzy topological spaces",*Fuzzy sets and System*, 88: 81-89.

Zadeh, L. A.,1965," Fuzzy Sets", *Information and Control* , 8 (3): 338-353.